King Saud University
College of Sciences
Department of Mathematics

Second Midterm Exam, S2-1443H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 minutes - Marks: 25

Answer the following questions.

Q1: [5+7]

(a) If $X \sim N(\theta, v)$, show that the normal distribution is a member of the linear exponential family.

(b) Let X have pdf of the transformed beta distribution, $f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$. Find the limiting distribution of X as $\tau \to \infty$.

Q2: [7+6]

(a) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the mean of an exponential model and the value of the log-likelihood function.

(b) A random sample of size 5 is taken from a Weibull distribution with $\tau = 2$. Two of the sample observations are known to exceed 50 and the three remaining observations are 20, 30, and 45. Determine the maximum likelihood estimate of θ.

The Model Answer

Q1: [5+7]

(a)

For $X \sim N(\theta, \upsilon)$

The pdf is given by

$$f(x;\theta) = (2\pi\upsilon)^{\frac{-1}{2}} exp[\frac{-1}{2\upsilon}(x-\theta)^2]$$

$$= (2\pi\upsilon)^{\frac{-1}{2}} exp[-\frac{x^2}{2\upsilon} + \frac{\theta}{\upsilon}x - \frac{\theta^2}{2\upsilon}]$$

$$f(x;\theta) = \frac{[(2\pi\upsilon)^{\frac{-1}{2}} \exp(-\frac{x^2}{2\upsilon})]\exp(\frac{\theta}{\upsilon}x)}{\exp(\frac{\theta^2}{2\upsilon})}$$

which of the form $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

where $p(x) = [(2\pi\upsilon)^{\frac{-1}{2}} \exp(-\frac{x^2}{2\upsilon})]$, $r(\theta) = \frac{\theta}{\upsilon}$ and $q(\theta) = \exp(\frac{\theta^2}{2\upsilon})$

$\therefore$ The normal distribution is a member of the linear exponential family.

(b)

$$\because\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad \text{(transformed beta pdf)}$$

let α be constant and $\theta\tau^{1/\gamma} \to \xi$

$\Rightarrow\ \theta = \xi\tau^{-1/\gamma}$

$$\therefore\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau}(1/\xi\tau^{1/\gamma})^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$= \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)(\xi\tau^{-1/\gamma})^{\gamma\tau}[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$\therefore\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}} \qquad (1)$$

$$\because \lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad \text{Stirling's formula}$$

$$\Rightarrow \Gamma(\tau) = e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2} \text{ as } \tau\to\infty \qquad (2)$$

$$\Rightarrow \Gamma(\alpha+\tau) = e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2} \text{ as } \alpha+\tau\to\infty \quad (3)$$

Substitute (2), (3) in (1)

$$\therefore f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)e^{-\tau}(\tau)^{\tau-1/2}(2\pi)^{1/2}\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\Rightarrow f(x) = \frac{e^{-\alpha}(\frac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\tau^{-\alpha}\tau^{-\tau}\xi^{\gamma(\tau+\alpha)}\xi^{-\gamma\alpha}x^{-\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\because \lim_{a\to\infty}(1+\tfrac{x}{a})^{a+b} = e^{x} \quad \therefore \lim_{\tau\to\infty}(\tfrac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2} = \lim_{\tau\to\infty}(1+\tfrac{\alpha}{\tau})^{\tau+\alpha-1/2} = e^{\alpha}$$

where $\lim_{\tau\to\infty}(1+\frac{\alpha}{\tau})^{\alpha} = 1$

$$\therefore f(x) = \frac{e^{-\alpha}e^{\alpha}\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}(\frac{1}{\tau})^{\alpha+\tau}(\frac{\xi}{x})^{\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore f(x) = \frac{\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}[\frac{1}{\tau}(\frac{\xi}{x})^{\gamma}]^{\alpha+\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore f(x) = \frac{\gamma(\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)x}\cdot\frac{1}{\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\because \lim_{\tau\to\infty}\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau} = e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x) = \frac{\gamma(\xi/x)^{\gamma\alpha}e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)x} \qquad \text{as } \tau\to\infty$$

which is the inverse transformed gamma pdf with parameters α, ξ and γ

Q2: [7+6]

(a)

The likelihood function is

$$L(\theta) = f(27)f(82)f(115)f(126)f(155)f(161)f(243)$$
$$= \theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\theta^{-1}e^{-115/\theta}\theta^{-1}e^{-126/\theta}\theta^{-1}e^{-155/\theta}\theta^{-1}e^{-161/\theta}\theta^{-1}e^{-243/\theta}$$
$$= \theta^{-7}e^{-909/\theta}$$

$\therefore l(\theta) = -7\ln\theta - 909\theta^{-1}$

which is known as log-likelihood function, to get the likelihood estimate of the parameter θ

Set $l^{'}(\theta) = 0$

$\Rightarrow -7\theta^{-1} + 909\theta^{-2} = 0$

$\therefore \hat{\theta} = 129.85714$ which is the MLE of the mean of an exponential model.

$\therefore l(\hat{\theta}) = -41.065$

(b)

For Weibull distribution

$F(x) = 1 - e^{-(x/\theta)^2}$, $f(x) = \frac{2x}{\theta^2} e^{-(x/\theta)^2}$

The likelihood function is

$$L(\theta) = f(20)f(30)f(45)[1 - F(50)]^2$$

$$\therefore L(\theta) = \frac{40}{\theta^2}e^{-(20/\theta)^2} \frac{60}{\theta^2}e^{-(30/\theta)^2} \frac{90}{\theta^2}e^{-(45/\theta)^2}\left[e^{-(50/\theta)^2}\right]^2$$

$$= 216{,}000\theta^{-6}e^{-8325/\theta^2}$$

$\therefore l(\theta) = -6\ln\theta - 8325\theta^{-2}$, by neglecting the constant term

which is known as log-likelihood function, to get $\hat{\theta}$ Set $l^{'}(\theta) = 0$

$\Rightarrow -6\theta^{-1} + 2(8325)\theta^{-3} = 0$

$\therefore 6\theta^2 = 16650$

$\therefore \hat{\theta} = \sqrt{\frac{16650}{6}} \simeq 52.68$