King Saud University
College of Sciences
Department of Mathematics

Second Midterm Exam, S2-1443H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 minutes - Marks: 25

Answer the following questions.

Q1: [5+7]

(a) If $X \sim N(\theta, v)$, show that the normal distribution is a member of the linear exponential family.

(b) Let X have pdf of the transformed beta distribution, $f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$. Find the limiting distribution of X as $\tau \to \infty$.

Q2: [7+6]

Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the parameter θ for exponential distribution, and for a gamma distribution with $\alpha = 2,$, and for Weibull distribution with $\tau = 2.$ Also, find the value of the log-likelihood function in each case.

The Model Answer

Q1: [5+7]

(a)

For $X \sim N(\theta,\upsilon)$
The pdf is given by

$$f(x;\theta)=(2\pi\upsilon)^{\frac{-1}{2}}\exp[\tfrac{-1}{2\upsilon}(x-\theta)^2]$$
$$=(2\pi\upsilon)^{\frac{-1}{2}}\exp[-\tfrac{x^2}{2\upsilon}+\tfrac{\theta}{\upsilon}x-\tfrac{\theta^2}{2\upsilon}]$$

$$f(x;\theta)=\frac{[(2\pi\upsilon)^{\frac{-1}{2}}\exp(-\frac{x^2}{2\upsilon})]\exp(\frac{\theta}{\upsilon}x)}{\exp(\frac{\theta^2}{2\upsilon})}$$

which of the form $f(x;\theta)=\dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

where $p(x)=[(2\pi\upsilon)^{\frac{-1}{2}}\exp(-\frac{x^2}{2\upsilon})]$, $r(\theta)=\frac{\theta}{\upsilon}$ and $q(\theta)=\exp(\frac{\theta^2}{2\upsilon})$

$\therefore$ The normal distribution is a member of the linear exponential family.

(b)

$$\because\ f(x)=\frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}\quad\text{(transformed beta pdf)}$$

let α be constant and $\theta\tau^{1/\gamma}\to\xi$

$\Rightarrow\ \theta=\xi\tau^{-1/\gamma}$

$$\therefore\ f(x)=\frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau}(1/\xi\tau^{1/\gamma})^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$
$$=\frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)(\xi\tau^{-1/\gamma})^{\gamma\tau}[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$\therefore\ f(x)=\frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}\qquad(1)$$

$$\because\ \lim_{\alpha\to\infty}\frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)}=1\qquad\text{Stirling's formula}$$

$$\Rightarrow\ \Gamma(\tau)=e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2}\ \text{ as }\ \tau\to\infty\qquad(2)$$

$$\Rightarrow\ \Gamma(\alpha+\tau)=e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\ \text{ as }\ \alpha+\tau\to\infty\qquad(3)$$

Substitute (2), (3) in (1)

$$\therefore\ f(x)=\frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)e^{-\tau}(\tau)^{\tau-1/2}(2\pi)^{1/2}\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\Rightarrow f(x)=\frac{e^{-\alpha}(\frac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\tau^{-\alpha}\tau^{-\tau}\xi^{\gamma(\tau+\alpha)}\xi^{-\gamma\alpha}x^{-\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\because\ \lim_{a\to\infty}(1+\tfrac{x}{a})^{a+b}=e^{x}\quad \therefore\ \lim_{\tau\to\infty}(\tfrac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2}=\lim_{\tau\to\infty}(1+\tfrac{\alpha}{\tau})^{\tau+\alpha-1/2}=e^{\alpha}$$

where $\lim_{\tau\to\infty}(1+\frac{\alpha}{\tau})^{\alpha}=1$

$$\therefore\ f(x)=\frac{e^{-\alpha}e^{\alpha}\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}(\frac{1}{\tau})^{\alpha+\tau}(\frac{\xi}{x})^{\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore\ f(x)=\frac{\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}[\frac{1}{\tau}(\frac{\xi}{x})^{\gamma}]^{\alpha+\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore f(x)=\frac{\gamma(\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)x}.\frac{1}{\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\because\ \lim_{\tau\to\infty}\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}=e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x)=\frac{\gamma(\xi/x)^{\gamma\alpha}e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)x}\quad \text{as } \tau\to\infty$$

which is the inverse transformed gamma pdf with parameters α, ξ and γ

Q2: [7+6]

(a)

The likelihood function is

$$\begin{aligned}L(\theta)&=f(27)f(82)f(115)f(126)f(155)f(161)f(243)\\&=\theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\theta^{-1}e^{-115/\theta}\theta^{-1}e^{-126/\theta}\theta^{-1}e^{-155/\theta}\theta^{-1}e^{-161/\theta}\theta^{-1}e^{-243/\theta}\\&=\theta^{-7}e^{-909/\theta}\end{aligned}$$

$$\therefore l(\theta)=-7\ln\theta-909\theta^{-1}$$

which is known as log-likelihood function, to get the likelihood estimate of the parameter θ

Set $l'(\theta)=0$

$\Rightarrow -7\theta^{-1} + 909\theta^{-2} = 0$

$\therefore \hat{\theta} = 129.85714$ which is the MLE of the mean of an exponential model.

$\therefore l(\hat{\theta}) = -41.065$

(b)

For Weibull distribution

$F(x) = 1 - e^{-(x/\theta)^2}, \; f(x) = \frac{2x}{\theta^2} e^{-(x/\theta)^2}$

The likelihood function is

$$L(\theta) = f(20)f(30)f(45)[1 - F(50)]^2$$

$$\therefore L(\theta) = \frac{40}{\theta^2} e^{-(20/\theta)^2} \frac{60}{\theta^2} e^{-(30/\theta)^2} \frac{90}{\theta^2} e^{-(45/\theta)^2} \left[e^{-(50/\theta)^2} \right]^2$$

$$= 216,000\theta^{-6} e^{-8325/\theta^2}$$

$$\therefore l(\theta) = -6\ln\theta - 8325\theta^{-2}, \text{ by neglecting the constant term}$$

which is known as log-likelihood function, to get $\hat{\theta}$ Set $l'(\theta) = 0$

$$\Rightarrow -6\theta^{-1} + 2(8325)\theta^{-3} = 0$$

$$\therefore 6\theta^2 = 16650$$

$$\therefore \hat{\theta} = \sqrt{\frac{16650}{6}} \simeq 52.68$$