King Saud University
College of Sciences
Department of Mathematics

Second Midterm Exam, S2-1447H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: [5+5]

(a) Let X have a Pareto distribution with parameters α and θ. Let $Y = \ln(1 + X/\theta)$. Identify the distribution of Y and its parameters.
(b) Let Λ have an exponential distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$, $x \geq 0$. Determine the unconditional or marginal distribution of X.

Q2: [5+5]

(a) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and the variance of the gamma distribution.
(b) Five losses are observed as 521, 658, 702, 819, and 1,217. Determine the maximum likelihood estimate of the parameter α of the single-parameter Pareto distribution, where its distribution function is defined as $F(x) = 1 - \left(\frac{500}{x}\right)^{\alpha}$, $x > 500$, $\alpha > 0$. Also, find the log-likelihood value.

Q3: [5]

Let X have pdf $f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$ (Transformed Beta)

Find the limiting distribution of X as $\tau \to \infty$ and $\theta\tau^{1/\gamma} \to \xi$, a constant. Identify this distribution.

The Model Answer

Q1: [5+5]

(a)

$\because X \sim Pareto\ (\alpha, \theta)$

$$\therefore F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

For $Y = \ln(1 + X/\theta),\ F_Y(y) = \Pr(Y \le y)$

$$F_Y(y) = \Pr[\ln(1 + X/\theta) \le y]$$

$$\therefore F_Y(y) = \Pr[1 + X/\theta \le e^y]$$

$$= \Pr[X \le \theta(e^y - 1)]$$

$$= 1 - \left[\frac{\theta}{\theta(e^y - 1) + \theta}\right]^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{e^y}\right)^{\alpha}$$

$$= 1 - e^{-\alpha y},$$

which is the distribution function of the exponential distribution with parameter $1/\alpha$.

(b)

$\because X|\Lambda \sim \text{weibull}(\lambda, \gamma),\ S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore\ A(x) = x^{\gamma}$

$\because \Lambda\ \sim\ \exp\ (\theta)$

$\therefore M_{\Lambda}(z) = (1 - \theta z)^{-1}$

$$\because\ S_X(x) = E[e^{-\Lambda A(x)}]$$

$$= M_{\Lambda}[-A(x)]$$

$$\therefore\ S_X(x) = M_{\Lambda}[-x^{\gamma}]$$

$$\therefore\ S_X(x) = (1 + \theta x^{\gamma})^{-1}$$

Which is a loglogistic distribution with the usual parameter θ replaced by $\theta^{\frac{-1}{\gamma}}$.

Q2: [5+5]

(a)

For $X \sim \text{gamma}(\alpha, \theta)$

$$\Rightarrow f(x;\theta)=\frac{\theta^{-\alpha}x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta)=\dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

$$=\frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^{\alpha}}$$

Where, $r(\theta)=-1/\theta$, $q(\theta)=\theta^{\alpha}$ and $p(x)=x^{\alpha-1}/\Gamma(\alpha)$

Hence, the gamma distribution is a member of the linear exponential family.

$\therefore$ The mean, $E(X)=\mu(\theta)=\dfrac{q'(\theta)}{r'(\theta)q(\theta)}$

$$=\frac{\alpha\theta^{\alpha-1}}{\theta^{-2}.\theta^{\alpha}}=\alpha\theta$$

and the variance,

$$Var(X)=v(\theta)=\frac{\mu'(\theta)}{r'(\theta)}$$

$$=\frac{\alpha}{1/\theta^{2}}=\alpha\theta^{2}$$

(b)

$\because X\sim$ single-parameter Pareto (α,θ)

$$F(x)=1-\left(\frac{500}{x}\right)^{\alpha},\quad x>500,\ \alpha>0.$$

$\therefore f(x)=\dfrac{\alpha(500)^{\alpha}}{x^{\alpha+1}}$, where $\theta=500$

The log-likelihood function is

$$\therefore \ln f(x|\alpha)=\ln\alpha+\alpha\ln 500-(\alpha+1)\ln x$$

$$\because l(\alpha)=\sum_{j=1}^{n}\ln f_{X_j}(x_j|\alpha)$$

$$\therefore l(\alpha)=\sum_{j=1}^{5}(\ln\alpha+\alpha\ln 500-(\alpha+1)\ln x_j)$$

$$=5\ln\alpha+5\alpha\ln 500-(\alpha+1)\sum_{j=1}^{5}\ln x_j$$

To get $\hat{\alpha}$, set $l'(\alpha)=0$

$$\Rightarrow \frac{5}{\alpha} + 5\ln 500 - \sum_{j=1}^{5} \ln x_j = 0$$

$$\therefore 5\alpha^{-1} + 5\ln 500 - 33.1111 = 0$$

$$\Rightarrow 5\alpha^{-1} - 2.0381 = 0$$

$$\therefore \hat{\alpha} = \frac{5}{2.0381} \simeq 2.45$$

$$\therefore l(\hat{\alpha}) \simeq -33.6239$$

Q3: [5]

$$\because\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad \text{(transformed beta pdf)}$$

let α be constant and $\theta\tau^{1/\gamma} \to \xi$

$$\Rightarrow\ \theta = \xi\tau^{-1/\gamma}$$

$$\therefore\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau}(1/\xi\tau^{1/\gamma})^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$= \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)(\xi\tau^{-1/\gamma})^{\gamma\tau}[1+x^{\gamma}\theta^{-\gamma}]^{\alpha+\tau}}$$

$$\therefore\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}} \quad (1)$$

$$\because\ \lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad \text{Stirling's formula}$$

$$\Rightarrow\ \Gamma(\tau) = e^{-\tau}\tau^{\tau-1/2}(2\pi)^{1/2} \text{ as } \tau\to\infty \quad (2)$$

$$\Rightarrow\ \Gamma(\alpha+\tau) = e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2} \text{ as } \alpha+\tau\to\infty \quad (3)$$

Substitute (2), (3) in (1)

$$\therefore\ f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\alpha)e^{-\tau}(\tau)^{\tau-1/2}(2\pi)^{1/2}\xi^{\gamma\tau}\tau^{-\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\Rightarrow f(x) = \frac{e^{-\alpha}(\frac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}x^{-\gamma\tau-\gamma\alpha}}{\Gamma(\alpha)\tau^{-\alpha}\tau^{-\tau}\xi^{\gamma(\tau+\alpha)}\xi^{-\gamma\alpha}x^{-\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\because\ \lim_{a\to\infty}(1+\tfrac{x}{a})^{a+b} = e^{x} \quad \therefore\ \lim_{\tau\to\infty}(\tfrac{\alpha+\tau}{\tau})^{\alpha+\tau-1/2} = \lim_{\tau\to\infty}(1+\tfrac{\alpha}{\tau})^{\tau+\alpha-1/2} = e^{\alpha}$$

where $\lim_{\tau\to\infty}(1+\frac{\alpha}{\tau})^{\alpha} = 1$

$$\therefore\ f(x) = \frac{e^{-\alpha}e^{\alpha}\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}(\frac{1}{\tau})^{\alpha+\tau}(\frac{\xi}{x})^{\gamma(\tau+\alpha)}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore\ f(x)=\frac{\gamma x^{-\gamma\alpha-1}}{\Gamma(\alpha)\xi^{-\gamma\alpha}[\frac{1}{\tau}(\frac{\xi}{x})^{\gamma}]^{\alpha+\tau}[1+x^{\gamma}\xi^{-\gamma}\tau]^{\alpha+\tau}}$$

$$\therefore f(x)=\frac{\gamma(\xi/x)^{\gamma\alpha}}{\Gamma(\alpha)x}.\frac{1}{\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}}$$

$$\because\ \lim_{\tau\to\infty}\left[1+\frac{(\xi/x)^{\gamma}}{\tau}\right]^{\alpha+\tau}=e^{(\xi/x)^{\gamma}}$$

$$\therefore f(x)=\frac{\gamma(\xi/x)^{\gamma\alpha}e^{-(\xi/x)^{\gamma}}}{\Gamma(\alpha)x}\quad \text{as } \tau\to\infty$$

which is the inverse transformed gamma pdf with parameters α, ξ and γ