King Saud University
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Department of Mathematics

Second Midterm Exam, S2-1446H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [3+5]

There are two types of drivers. Good drivers make up 75% of the population and in one year have zero claims with probability 0.8, one claim with probability 0.1, and two claims with probability 0.1. Bad drivers make up the other 25% of the population and have zero, one, or two claims with probabilities 0.6, 0.2, and 0.2, respectively.

(a) Describe this process by using the concept of the risk parameter Θ.

(b) For a particular policyholder, suppose that we have observed $x_1 = 0$ and $x_2 = 1$ for past claims. Determine the posterior distribution of $\Theta | X_1 = 0,\ X_2 = 1$ and the predictive distribution of $X_3 | X_1 = 0,\ X_2 = 1$.

Q2: [6+3]

(a) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimates of the parameter θ for the inverse exponential and inverse gamma with $\alpha = 2$ distributions. Also, find the value of the log-likelihood function in each case.

Hint: For inverse gamma distribution $f(x;\theta) = \dfrac{(\theta / x)^{\alpha} e^{-\theta/x}}{x\,\Gamma(\alpha)}$.

(b) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$, $x \geq 0$. Determine the unconditional or marginal distribution of X.

Q3: [4+4]

Suppose that there were 10 observations of claims with five being zero and others being 253, 398, 439, 129, 627. Let the manual premium $M = 210$ and assume that the number of claims has a Poisson distribution. Determine the full credibility and partial credibility according to the average number of claims. Use $r = 0.05$ and $p = 0.95$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+5]

(a)

x	$\Pr(X = x \mid \Theta = G)$	$\Pr(X = x \mid \Theta = B)$	θ	$\Pr(\Theta = \theta)$
0	0.8	0.6	G	0.75
1	0.1	0.2	B	0.25
2	0.1	0.2		

(b)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G \mid 0,1) = \frac{f(0 \mid G) f(1 \mid G) \pi(G)}{f_X(0,1)}$$

where $f_X(0,1) = \sum_{\theta} f_{X_1|\Theta}(0 \mid \theta) f_{X_2|\Theta}(1 \mid \theta) \pi(\theta)$

$$\begin{aligned} f_X(0,1) &= 0.8(0.1)(0.75) + 0.6(0.2)(0.25) \\ &= 0.09 \end{aligned}$$

$$\pi(G \mid 0,1) = \frac{0.8(0.1)(0.75)}{0.09} \simeq 0.67$$

$$\pi(B \mid 0,1) = \frac{0.6(0.2)(0.25)}{0.09} \simeq 0.33$$

For the predictive distribution, the predictive probabilities are given by

$$\begin{aligned} f_{X_3|X}(0 \mid 0,1) &= \sum_{\theta} f(0 \mid \theta) \pi(\theta \mid 0,1) \\ &= f(0 \mid G)\pi(G \mid 0,1) + f(0 \mid B)\pi(B \mid 0,1) \\ &= 0.8(0.67) + 0.6(0.33) \\ &= 0.734, \end{aligned}$$

$$\begin{aligned} f_{X_3|X}(1 \mid 0,1) &= \sum_{\theta} f(1 \mid \theta) \pi(\theta \mid 0,1) \\ &= f(1 \mid G)\pi(G \mid 0,1) + f(1 \mid B)\pi(B \mid 0,1) \\ &= 0.1(0.67) + 0.2(0.33) \\ &= 0.133, \end{aligned}$$

and $f_{X_3|X}(2 \mid 0,1) = \sum_{\theta} f(2 \mid \theta) \pi(\theta \mid 0,1)$

$$\begin{aligned} &= f(2 \mid G)\pi(G \mid 0,1) + f(2 \mid B)\pi(B \mid 0,1) \\ &= 0.1(0.67) + 0.2(0.33) \\ &= 0.133. \end{aligned}$$

Q2: [6+3]

(a)

(1) For inv. exponential distribution, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{x_j^2}$$

$$l(\theta) = \sum_{j=1}^{n} \ln\theta - \sum_{j=1}^{n} \theta / x_j - 2\sum_{j=1}^{n} \ln x_j$$

$$l(\theta) = n\ln\theta - \theta\sum_{j=1}^{n} x_j^{-1} - 2\sum_{j=1}^{n} \ln x_j$$

$\therefore\ l(\theta) = n\ln\theta - ny\theta - 2\sum_{j=1}^{n} \ln x_j$, where $y = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

(2) To get max. estimate of θ (i.e. $\hat{\theta}$), set $l'(\theta) = 0$

$$l'(\theta) = n\theta^{-1} - ny = 0$$

$\Rightarrow\ \theta^{-1} = y$ i.e. $\hat{\theta} = 1/y$

$\therefore\ \hat{\theta} \approx 84.7$

(3) The value of the log-likelihood function for inv. exponential distribution is given by

$$l(\hat{\theta}) = 7\ln\hat{\theta} - 7 - 2\sum_{j=1}^{7} \ln x_j$$

$$= 7\ln(84.702347) - 7 - 2(32.90166107)$$

$$\approx -41.73$$

(4) For inv. gamma distribution with $\alpha = 2$, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \theta^2 x_j^{-3} e^{-\theta/x_j}$$

$l(\theta) = 2\sum_{j=1}^{n} \ln\theta - 3\sum_{j=1}^{n} \ln x_j - ny\theta$, where $y = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

$$l(\theta) = 2n\ln\theta - ny\theta - 3\sum_{j=1}^{n} \ln x_j$$

(5) To get $\hat{\theta}$, let $l'(\theta) = 0$

$$l'(\theta) = 2n\theta^{-1} - ny = 0$$

$\Rightarrow\ \theta^{-1} = y/2$ i.e. $\hat{\theta} = 2/y$

$\therefore\ \hat{\theta} \approx 169.4$

(6) The value of the log-likelihood function for inv. gamma distribution is given by

$$l(\hat{\theta}) = 14\ln\hat{\theta} - 14 - 3\sum_{j=1}^{7}\ln x_j$$

$$= 14\ln(169.404694) - 14 - 3(32.90166107)$$

$$\approx -\ 40.85$$

(b)

$\because X|\Lambda \sim$ weibull(λ,γ), $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore\ A(x) = x^{\gamma}$

$\because \Lambda\ \sim$ gamma (α,θ)

$\therefore M_{\Lambda}(z) = (1-\theta z)^{-\alpha}$

$\because\ S_X(x) = E[e^{-\Lambda A(x)}]$

$= M_{\Lambda}[-A(x)]$

$\therefore\ S_X(x) = M_{\Lambda}[-x^{\gamma}]$

$\therefore\ S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with the usual parameter θ replaced by $\theta^{\frac{-1}{\gamma}}$.

Q3: [4+4]

1. For full credibility

at $p = 0.95$, $\Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

$$n \ge \lambda_0\left(\frac{\sigma^2}{\xi^2}\right)$$

$$n \ge \lambda_0\left(\frac{\lambda}{\lambda^2}\right)$$

$$n \ge \left(\frac{\lambda_0}{\lambda}\right)$$

$\because \lambda_0 = 1536.64$ and the expected number of claims is $\lambda = 0.5$

$\therefore n \ge 3073.28$

2. For partial credibility

The credibility factor is $Z = \sqrt{\frac{10}{3073.28}} \approx 0.057$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$
$$= 0.057(184.6) + (1-0.057)(210)$$
$$\therefore P_c \approx 208.55$$