King Saud University
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Department of Mathematics

Second Midterm Exam, S1-1447H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [4+5]

(a) Losses in 2023 follow the density function $f(x) = 2x^{-3}, x \geq 1$, where x is the loss in millions of dollars. Inflation of 10% impacts all claims uniformly from 2023 to 2024. Determine the cdf of losses for 2024 and use it to determine the probability that a 2024 loss exceeds 2,200,000.

(b) Consider the exponential-inverse Gaussian frailty model with

$$a(x) = \frac{\theta}{2\sqrt{1+\theta x}}, \ \theta > 0$$

(i) Determine the conditional survival function $S_{X|\Lambda}(x|\lambda)$.

(ii) If Λ has a gamma distribution with parameters $\theta = 1$ and α replaced by 2α, determine the marginal or unconditional survival function of X and identify it.

Q2: [4+4]

(a) Let X have pdf $f(x) = \dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$ (Transformed Beta)

Find the pdf as $\theta \to \infty$, $\alpha \to \infty$, and $\theta/\alpha^{1/\gamma} \to \xi$, a constant. Identify this distribution.

(b) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimates of the parameter θ for the inverse exponential and inverse gamma with $\alpha = 2$ distributions. Also, find the value of the log-likelihood function in each case.

Hint: For inverse gamma distribution $f(x;\theta) = \dfrac{(\theta/x)^{\alpha} e^{-\theta/x}}{x\,\Gamma(\alpha)}$.

Q3: [4+4]

Suppose that there were 10 observations of claims with five being zero and others being 253, 398, 439, 129, 627. Let the manual premium $M = 210$ and assume that the number of claims has a Poisson distribution. Determine the full credibility and partial credibility according to the average number of claims. Use $r = 0.05$ and $p = 0.95$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+5]

(a)

$$F(x) = \int_1^x 2t^{-3}\, dt$$

$$= 1 - x^{-2}$$

$\because Y = 1.1\, X$

$$F_Y(y) = 1 - \left(\frac{y}{1.1}\right)^{-2}$$

$\therefore \Pr(Y > 2.2) = 1 - F_Y(2.2) = 0.25$

(b)

(i)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\, dt$$

$$= \int_0^x \frac{\theta}{2\sqrt{1+\theta t}} dt$$

$$= \frac{1}{2}\int_0^x (1+\theta t)^{-\frac{1}{2}} \theta dt$$

$\therefore A(x) = \sqrt{1+\theta x} - 1$

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$

$$= e^{-\lambda(\sqrt{1+\theta x}-1)}$$

(ii)

$\because \Lambda \sim gamma\ (2\alpha, 1)$

$\therefore$ The moment generating function of the frailty random variable Λ is

$$M_\Lambda(z) = E(e^{z\Lambda})$$

$$= \left(\frac{1}{1-z}\right)^{2\alpha} = (1-z)^{-2\alpha}$$

The marginal survival function is

$$S_X(x) = E(e^{-\Lambda A(x)})$$

$$= M_\Lambda[-A(x)]$$

$$\therefore S_X(x) = (1 + \sqrt{1+\theta x} - 1)^{-2\alpha}$$

$$= (1+\theta x)^{-\alpha}$$

Which is a Pareto distribution.

Q2: [4+4]

(a)

$$\because\ f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^\gamma]^{\alpha+\tau}} \text{ (transformed beta pdf) } (1)$$

let $\theta = \xi\alpha^{1/\gamma}$ where $\theta/\alpha^{1/\gamma} = \xi = \text{constant}$

When $\theta \to \infty$, $\alpha \to \infty$ and $\theta/\alpha^{1/\gamma} \to \xi$, a constant

By using Stirling's formula

$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \qquad (2)$$

$\Rightarrow$

$$\lim_{\alpha+\tau\to\infty} \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)} = 1 \qquad (3)$$

Substitute (2) and (3) in (1)

$$f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}\gamma x^{\gamma\tau-1}}{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}\Gamma(\tau)(\xi\alpha^{1/\gamma})^{\gamma\tau}(1+x^{\gamma}\xi^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

where $\theta = \xi\alpha^{1/\gamma} \Rightarrow \theta^{-\gamma} = \xi^{-\gamma}\alpha^{-1}$

$$\therefore f(x) = \frac{e^{-\tau}\left(\frac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-1/2}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\xi^{\gamma\tau}\left(1+\frac{(x/\xi)^{\gamma}}{\alpha}\right)^{\alpha+\tau}}$$

$$\because \lim_{\alpha\to\infty}\left(\tfrac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-1/2} = \lim_{\alpha\to\infty}\left(1+\tfrac{\tau}{\alpha}\right)^{\alpha+\tau-1/2} = e^{\tau}$$

where $\lim_{\alpha\to\infty}\left(1+\tfrac{\tau}{\alpha}\right)^{\tau-1/2} = 1$, $\lim_{\alpha\to\infty}\left(1+\tfrac{\tau}{\alpha}\right)^{\alpha} = e^{\tau}$

and $\lim_{\alpha\to\infty}\left(1+\tfrac{(x/\xi)^{\gamma}}{\alpha}\right)^{\alpha+\tau} = e^{(x/\xi)^{\gamma}}$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{\gamma x^{\gamma\tau-1}e^{-(x/\xi)^{\gamma}}}{\Gamma(\tau)\xi^{\gamma\tau}}$$

which is the pdf of the transformed gamma distribution with parameters τ, ξ and γ.

(b)

(1) For inv. exponential distribution, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{x_j^2}$$

$$l(\theta) = \sum_{j=1}^{n}\ln\theta - \sum_{j=1}^{n}\theta/x_j - 2\sum_{j=1}^{n}\ln x_j$$

$$l(\theta) = n\ln\theta - \theta\sum_{j=1}^{n}x_j^{-1} - 2\sum_{j=1}^{n}\ln x_j$$

$$\therefore\ l(\theta) = n\ln\theta - ny\theta - 2\sum_{j=1}^{n}\ln x_j,\ \text{ where } y = \frac{1}{n}\sum_{j=1}^{n}x_j^{-1}$$

(2) To get max. estimate of θ (i.e. $\hat{\theta}$), set $l'(\theta) = 0$

$$l'(\theta) = n\theta^{-1} - ny = 0$$

$\Rightarrow \ \theta^{-1} = y$ i.e. $\hat{\theta} = 1/y$

$\therefore \ \hat{\theta} \approx 84.7$

(3) The value of the log-likelihood function for inv. exponential distribution is given by

$$l(\hat{\theta}) = 7\ln\hat{\theta} - 7 - 2\sum_{j=1}^{7}\ln x_j$$

$$= 7\ln(84.702347) - 7 - 2(32.90166107)$$

$$\approx -41.73$$

(4) For inv. gamma distribution with $\alpha = 2$, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n}\theta^2 x_j^{-3} e^{-\theta/x_j}$$

$$l(\theta) = 2\sum_{j=1}^{n}\ln\theta - 3\sum_{j=1}^{n}\ln x_j - ny\theta, \text{ where } y = \frac{1}{n}\sum_{j=1}^{n}x_j^{-1}$$

$$l(\theta) = 2n\ln\theta - ny\theta - 3\sum_{j=1}^{n}\ln x_j$$

(5) To get $\hat{\theta}$, let $l'(\theta) = 0$

$$l'(\theta) = 2n\theta^{-1} - ny = 0$$

$\Rightarrow \ \theta^{-1} = y/2$ i.e. $\hat{\theta} = 2/y$

$\therefore \ \hat{\theta} \approx 169.4$

(6) The value of the log-likelihood function for inv. gamma distribution is given by

$$l(\hat{\theta}) = 14\ln\hat{\theta} - 14 - 3\sum_{j=1}^{7}\ln x_j$$

$$= 14\ln(169.404694) - 14 - 3(32.90166107)$$

$$\approx -40.85$$

Q3: [4+4]

1. For full credibility

at $p = 0.95$, $\Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p/r)^2 = (1.96/0.05)^2 = 1536.64$

$$n \geq \lambda_0 \left(\frac{\sigma^2}{\xi^2} \right)$$

$$n \geq \lambda_0 \left(\frac{\lambda}{\lambda^2} \right)$$

$$n \geq \left(\frac{\lambda_0}{\lambda} \right)$$

$\because \lambda_0 = 1536.64$ and the expected number of claims is $\lambda = 0.5$

$\therefore n \geq 3073.28$

2. For partial credibility

The credibility factor is $Z = \sqrt{\frac{10}{3073.28}} \approx 0.057$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$
$$= 0.057(184.6) + (1-0.057)(210)$$

$\therefore P_c \approx 208.55$