King Saud University
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Department of Mathematics

Second Midterm Exam, S1-1446H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [4+4]

(a) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table.

Year	1	2	3
Claims	450	550	500

Determine the full credibility standard and then determine the net premium for next year's claims assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

(b) Let X have a Pareto distribution with parameters α and θ. Let $Y = \ln(1 + X/\theta)$. Determine the distribution of Y and its parameters (give its name).

Q2: [5+4]

(a) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the parameter θ for **inverse** exponential distribution, and find the value of the log-likelihood function.

Hint: $f(x;\theta) = \dfrac{\theta e^{-\theta/x}}{x^2}$ for inverse exponential distribution.

(b) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$, $x \geq 0$. Determine the unconditional or marginal distribution of X.

Q3: [4+4]

Suppose that there were 10 observations of claims with five being zero and others being 253, 398, 439, 129, 627. Let the manual premium $M = 210$ and assume that the number of claims has a Poisson distribution. Determine the full credibility and partial credibility according to the average total payment. Use $r = 0.05$ and $p = 0.95$.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+4]

(a)

at $p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

The mean is $\xi = E(X_j)$

$$= \frac{450+550+500}{3} = 500$$

and variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)}{n-1}$

$$= \frac{50^2 + 50^2 + 0^2}{2} = 2500$$

$\Rightarrow \sigma = \sqrt{2500} = 50$

For full credibility

$$n \ge \lambda_0 (\tfrac{\sigma}{\xi})^2$$

$$n \ge 1536.64(\tfrac{50}{500})^2$$

$$n \ge 15.3664$$

The credibility factor is $Z = \sqrt{\frac{n}{\lambda_0 \sigma^2/\xi^2}}$

$$= \sqrt{\tfrac{3}{15.3664}} = 0.44185$$

The partial credibility through premium is

$P_c = Z\overline{X} + (1-Z)M$

$= 0.44185(500) + (1-0.44185)(600)$

$\simeq 556$

(b)

$\because X \sim Pareto\ (\alpha, \theta)$

$\therefore F_X(x) = 1 - \left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$

For $Y = \ln(1 + X/\theta),\ F_Y(y) = \Pr(Y \le y)$

$F_Y(y) = \Pr[\ln(1 + X/\theta) \le y]$

$$\therefore F_Y(y) = \Pr[1 + X/\theta \le e^y]$$

$$= \Pr[X \le \theta(e^y - 1)]$$

$$= 1 - \left[\frac{\theta}{\theta(e^y - 1) + \theta}\right]^\alpha$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{e^y}\right)^\alpha$$

$$= 1 - e^{-\alpha y},$$

which is the exponential distribution function with parameter $1/\alpha$.

Q2: [5+4]

(a)

The likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{x_j^2}$$

The log-likelihood function is

$$l(\theta) = n \ln \theta - \theta \sum_{j=1}^{n} x_j^{-1} - 2 \sum_{j=1}^{n} \ln x_j$$

$$\therefore l(\theta) = n \ln \theta - ny\theta - 2 \sum_{j=1}^{n} \ln x_j \text{, where } y = \frac{1}{n} \sum_{j=1}^{n} x_j^{-1}$$

To get the maximum estimate of θ (i.e. $\hat{\theta}$), let $l'(\theta) = 0$

$$\Rightarrow l'(\theta) = n\theta^{-1} - ny = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{y} \therefore \hat{\theta} = \frac{1}{0.0118} \simeq 84.7$$

$$\Rightarrow l(\theta) = n \ln \theta - ny(\tfrac{1}{y}) - 2 \sum_{j=1}^{n} \ln x_j$$

$$= n \ln \theta - n - 2 \sum_{j=1}^{n} \ln x_j$$

$$\therefore l(\hat{\theta}) = 7 \ln(84.7) - 7 - 2(32.9) \simeq -41.7$$

(b)

$$\because X|\Lambda \sim \text{weibull}(\lambda, \gamma), \;\; S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^\gamma}$$

$$\therefore \;\; A(x) = x^\gamma$$

$$\because \Lambda \;\sim\; \text{gamma}\,(\alpha, \theta)$$

$$\therefore M_\Lambda(z) = (1 - \theta z)^{-\alpha}$$

$$\because \;\; S_X(x) = E[e^{-\Lambda A(x)}]$$

$$= M_\Lambda[-A(x)]$$

$$\therefore \;\; S_X(x) = M_\Lambda[-x^\gamma]$$

$\therefore \; S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with the usual parameter θ replaced by $\theta^{\frac{-1}{\gamma}}$.

Q3: [4+4]

1. For full credibility

at $p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

$$n \geq \frac{\lambda_0}{\lambda}\left[1+\left(\frac{\sigma_Y}{\theta_Y}\right)^2\right],\ \text{where } \lambda_0 = 1536.64$$

and the expected number of claims is $\lambda = 0.5$

The mean is $\theta_Y = \dfrac{253+398+439+129+627}{5} = 369.2,$

variance is $\sigma_Y^2 = \dfrac{\sum_j (y_j - 369.2)^2}{4} = 35840.2$

$$n \geq \frac{1536.64}{0.5}\left[1+\frac{35840.2}{(369.2)^2}\right]$$

$$n \geq 3073.28\left[1+\frac{35840.2}{(369.2)^2}\right]$$

$\therefore$ For full credibility $n \geq 3881.35$

2. For partial credibility

The credibility factor is $Z = \sqrt{\dfrac{10}{3881.35}} = 0.050758$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$

$$= 0.050758(184.6) + (1-0.050758)(210)$$

$\therefore P_c = 208.71$