King Saud University
College of Sciences
Department of Mathematics

Midterm Exam, S2-1444H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 minutes - Marks: 30

Answer the following questions.

Q1: [4+4+4]

An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses: 100(0.4), 500(0.2), 1000(0.2), 2500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for the excess loss and left censored and shifted variables, where the deductible is set at 750. Also, calculate the probability mass function and the expected value of the limited loss variable (right censored variable) with a limit of 750.

Q2: [5+5]

(a) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and the variance of the gamma distribution.

(b) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

Q3: [8]

Five losses are observed as 521, 658, 702, 819, and 1,217. Determine the maximum likelihood estimate of the parameter α of the single-parameter Pareto distribution, where its distribution function is defined as

$$F(x) = 1 - \left(\frac{500}{x}\right)^{\alpha}, \quad x > 500, \ \alpha > 0.$$

Also, find the value of the log-likelihood function.

The Model Answer

Q1: [4+4+4]

The probability of exceeding the deductible is

$$\Pr(X > 750) = 1 - F(750),\ d = 750$$
$$= 0.2 + 0.1 + 0.1 = 0.4$$

For the excess loss variable $Y^p = X - d$, we have

$X_j - d$:	250	1750	9250
$\frac{p(x_j)}{1-F(d)}$:	0.2/0.4	0.1/0.4	0.1/0.4
	0.5	0.25	0.25

The expected value for the excess loss variable is defined as

$$e_X(d) = E(Y^p) = E(X - d \mid X > d)$$

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2{,}875$$

For the left censored and shifted variable $Y^L = (X-d)_+$

$(X_j - d)_+$:	0	250	1750	9250
$p(x_j)$:	0.6	0.2	0.1	0.1

The expected value for the left censored and shifted variable is

$$E(Y^L) = E[(X-d)_+],\ \text{where } Y^L = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases}$$

$$\therefore E(Y^L) = E[(X-d)_+]$$
$$= 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$$
$$= 1{,}150$$

For the limited loss variable (right censored variable)

$X_j \wedge d$:	100	500	750
$p(x_j)$:	0.4	0.2	0.4

The expected value for the limited loss variable (right censored variable) is

$$E(Y) = E(X \wedge d), \text{ where } Y = \begin{cases} X, & X < d \\ d, & X \geq d \end{cases}$$

$$\therefore E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$$
$$= 440$$

Not that:

$$E(X) = E[(X-d)_+] + E(X \wedge d)$$
$$= 1{,}150 + 440$$
$$= 1{,}590$$

Q2: [5+5]

(a)

For $X \sim \text{gamma}(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x) e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^{\alpha}}$$

Where, $r(\theta) = -1/\theta$, $q(\theta) = \theta^{\alpha}$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$$\therefore \text{The mean, } E(X) = \mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$$

$$= \frac{\alpha\theta^{\alpha-1}}{1/\theta^2.\theta^{\alpha}} = \alpha\theta$$

and the variance, $Var(X) = v(\theta)$

$$= \frac{\mu'(\theta)}{r'(\theta)}$$

$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

(b)

let $\Lambda \sim$ gamma (θ, α), $X|\Lambda \sim$ weibull(λ, γ)

$\therefore \; S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore \; A(x) = x^{\gamma}$

$\because \; S_X(x) = E[e^{-\Lambda A(x)}]$

$\qquad = M_{\Lambda}[-A(x)]$

$\therefore \; S_X(x) = M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z) = (1-\theta z)^{-\alpha}$

$\therefore \; S_X(x) = (1+\theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\theta \to \theta^{\frac{-1}{\gamma}}, \; \alpha \to \alpha$

Q3: [8]

$\because X \sim$ single-parameter Pareto (α, θ)

$$F(x) = 1 - \left(\frac{500}{x}\right)^{\alpha}, \quad x > 500, \; \alpha > 0.$$

$$\therefore f(x) = \frac{\alpha(500)^{\alpha}}{x^{\alpha+1}}, \text{ where } \theta = 500$$

The log-likelihood function is

$$\therefore \ln f(x|\alpha) = \ln\alpha + \alpha \ln 500 - (\alpha+1)\ln x$$

$$\because l(\alpha) = \sum_{j=1}^{n} \ln f_{X_j}(x_j|\alpha)$$

$$\therefore l(\alpha) = \sum_{j=1}^{5} (\ln\alpha + \alpha \ln 500 - (\alpha+1)\ln x_j)$$

$$= 5\ln\alpha + 5\alpha \ln 500 - (\alpha+1)\sum_{j=1}^{5} \ln x_j$$

To get $\hat{\alpha}$, set $l^{'}(\alpha) = 0$

$$\Rightarrow \frac{5}{\alpha} + 5\ln 500 - \sum_{j=1}^{5} \ln x_j = 0$$

$$\therefore 5\alpha^{-1} + 5\ln 500 - 33.1111 = 0$$

$$\Rightarrow 5\alpha^{-1} - 2.0381 = 0$$

$$\therefore \hat{\alpha} = \frac{5}{2.0381} \simeq 2.45$$

$$\therefore l(\hat{\alpha}) \simeq -33.6239$$