King Saud University
College of Sciences
Department of Mathematics

First Mid-Term Exam, S2-1447H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [3+4]

(a) For the model of automobile bodily injury claim that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-\left(\dfrac{2000}{x+2000}\right)^3, & x\geq 0.\end{cases}$$

Determine the survival, density, and hazard rate functions.

(b) The cdf of a random variable X is $F(x)=1-\exp\left(-\dfrac{x}{\theta}\right)$, $x>0$.

Find $e_X(x)$ and $E(X\wedge x)$.

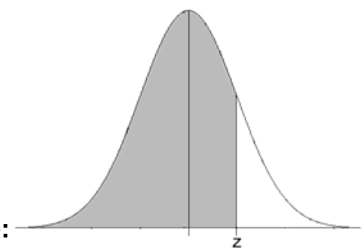
Q2: [5+4]

(a) The severities of individual claims have a Pareto distribution with parameters $\alpha = 8/3$ and $\theta = 8{,}000$. Use the central limit theorem to approximate the probability that the sum of 100 independent claims will exceed 700,000.
(b) Demonstrate that the Weibull distribution is a scale distribution.
Hint: For Weibull distribution, $F(x) = 1 - e^{-(x/\theta)^\tau}$.

Q3: [3+3+3]

Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed, and inverse transformed exponential distributions. Clarify the names of distributions.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+4]

(a) The survival function is

$$S(x)=1-F(x)$$

$$\therefore\ S(x)=\left(\frac{2000}{x+2000}\right)^3,\ x\geq 0$$

The density function is

$$f(x)=F^{'}(x)=-S^{'}(x)$$

$$\therefore\ f(x)\ =\frac{3(2000)^3}{(x+2000)^4},\ x>0$$

The hazard rate function

$$h(x)=\frac{f(x)}{S(x)}$$

$$\therefore\ h(x)=\frac{3}{(x+2000)},\ x>0$$

(b)

The mean excess loss function is

$$e_X(x)=\frac{\int_x^{\infty}S(t)dt}{S(x)}$$

$$\because S(x)=\exp\left(-\frac{x}{\theta}\right)$$

$$\Rightarrow e_X(x)=\frac{\int_x^{\infty}\exp\left(-\frac{t}{\theta}\right)dt}{\exp\left(-\frac{x}{\theta}\right)}$$

$$\therefore e_X(x)=\frac{-\theta.\exp\left(-\frac{t}{\theta}\right)\Big|_x^{\infty}}{\exp\left(-\frac{x}{\theta}\right)}=\theta$$

The limited (right censored) expected value function is

$$\because E(X\wedge x)=\ E(X)-e(x)S(x)$$

$$\therefore E(X\wedge x)=\theta-\theta.\exp\left(-x/\theta\right)$$

$$=\theta(1-e^{-x/\theta})$$

or simply

$$\because E(X\wedge x)=-\int_{-\infty}^{0}F(t)dt+\int_{0}^{x}s(t)dt$$

$$\therefore E(X \wedge x) = \int_0^x s(t)dt \text{ where } \int_{-\infty}^0 F(t)dt = 0$$

$$= \int_0^x \exp\left(-\frac{t}{\theta}\right)dt$$

$$= -\theta[e^{-t/\theta}]_0^x$$

$$\therefore E(X \wedge x) = \theta(1 - e^{-x/\theta})$$

Q2: [5+4]

(a)

$\because X = \text{Pareto}(8/3, 8000)$

The K^{th} moment is given by $\mathrm{E}(X^k) = \frac{\theta^k k!}{(\alpha-1)\ldots(\alpha-k)}$

$\therefore \mathrm{E}(X) = \mu = \frac{\theta}{(\alpha-1)} = \frac{8{,}000}{\frac{8}{3}-1}$ =4,800 and $\mathrm{E}(X^2) = \frac{\theta^2 2!}{(\alpha-1)(\alpha-2)} = 115{,}200{,}000$

$\therefore \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$

$= 115{,}200{,}000 - 4{,}800^2$

$= 92{,}160{,}000$

For independent random variables $X_1, X_2, \ldots, X_{100}$, the sum is

$S_{100} = X_1 + X_2 + \cdots + X_{100}$

$\therefore$ by using central limit theorem

$E(S_{100}) = 100(4800) = 480{,}000$ and

$Var(S_{100}) = 100(92{,}160{,}000) = 9{,}216{,}000{,}000$

The standard deviation for the sum S_{100} is $\sqrt{Var(S_{100})} = 96{,}000$

$$\therefore \Pr(S_{100} > 700{,}000) = 1 - \Phi\left(\frac{700{,}000 - 480{,}000}{96{,}000}\right) = 1 - \Phi(2.29) \cong 0.011$$

(b)

For the Weibull distribution, $X \sim$ Weibull (θ, τ)

and the distribution function is $F_X(x) = 1 - e^{-(x/\theta)^\tau}$

let $Y = cX$, $c > 0$, then

$F_Y(y) = pr(Y \le y)$

$= pr(X \le \frac{y}{c})$

$i.e.\ F_Y(y) = F_X(\frac{y}{c})$

$\therefore\ F_Y(y) = 1 - e^{-(y/c\theta)^\tau}$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore$ θ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution.

Q3: [3+3+3]

We have, $F_X(x) = 1 - e^{-x}$ (exp, dist. with no scale parameter), so we could obtain the following:

(1) The inverse exponential distribution with no scale parameter (where $\tau = -1$) has cdf

$F_Y(y) = 1 - F_X(y^{-1})$ Theorem

$= 1 - [1 - e^{-1/y}]$

$F_Y(y) = e^{-1/y}$

With the scale parameter added, it is $F(y) = e^{-\theta/y}$ (inverse exponential distribution)

(2) The transformed exponential distribution with no scale parameter (where $\tau > 0$) has cdf

$F_Y(y) = F_X(y^{\tau}),$ $\tau>0$ Theorem

$= 1 - e^{-y^{\tau}}$

$F_Y(y) = 1 - exp(-y^{\tau})$

With the scale parameter added, it is $F(y) = 1 - \exp[-(y/\theta)^{\tau}]$ (Weibull distribution)

(3) The inverse transformed exponential distribution with no scale parameter has cdf

$F_Y(y) = 1 - F_X(y^{-\tau})$ Theorem for negative τ

$= 1 - [1 - exp(-y^{-\tau})]$

$F_Y(y) = exp(-y^{-\tau})$

With the scale parameter added, it is

$F_Y(y) = exp[-(y/\theta)^{-\tau}] = exp[-(\theta/y)^{\tau}]$ (inverse Weibull distribution).