King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S2-1446H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [4+4]

If the random variable X has probability density function $f(x)=(1+2x^2)e^{-2x},\ x \geq 0.$

(a) Determine the survival and hazard rate functions.

(b) Find the mean excess loss function $e_X(x)$ and limited expected value function $E(X \wedge x)$.

Q2: [4+4]

(a) The severities of individual claims have a Pareto distribution with parameters $\alpha = 8/3$ and $\theta = 8{,}000$. Use the central limit theorem to approximate the probability that the sum of 100 independent claims will exceed 700,000.

(b) Let X have cdf $F_X(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Let $Y=\ln(1+X/\theta)$. Determine the distribution of Y and its parameters (give its name).

Q3: [3+3+3]

Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed, and inverse transformed exponential distributions. Clarify the names of distributions.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+4]

(a)

The survival function is

$$S(x) = \int_x^\infty (1+2t^2)e^{-2t}dt$$

$$= -\tfrac{1}{2}e^{-2t}\Big|_x^\infty + 2I, \text{ where } I = \int_x^\infty t^2 e^{-2t}dt$$

$$I = \int t^2 e^{-2t}dt$$

$$= -\tfrac{1}{4}e^{-2t} - \tfrac{t}{2}e^{-2t} - \tfrac{t^2}{2}e^{-2t}$$

$$\therefore\ S(x) = -(1+t+t^2)e^{-2t}\Big|_x^\infty$$

$$= (1+x+x^2)e^{-2x},\ x \ge 0$$

The hazard rate function is

$$h(x) = -\frac{d}{dx}[\ln S(x)]$$

$$\because S(x) = (1+x+x^2)e^{-2x},$$

$$\ln S(x) = -2x + \ln(1+x+x^2)$$

$$\therefore\ h(x) = 2 - \frac{1+2x}{1+x+x^2}$$

or simply,

$$h(x) = \frac{f(x)}{S(x)} = \frac{1+2x^2}{1+x+x^2}$$

(b)

The mean excess loss function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)} \qquad (1)$$

From (i) $S(x) = (1+x+x^2)e^{-2x}$ (2)

We can deduce that,

$$\int_x^\infty S(t)dt = \int_x^\infty (1+t+t^2)e^{-2t}dt$$

$$= -(1+t+\tfrac{1}{2}t^2)e^{-2t}\Big|_x^\infty$$

$$= (1+x+\tfrac{1}{2}x^2)e^{-2x} \qquad (3)$$

Where $I = \int t^2 e^{-2t}dt = -\tfrac{1}{4}e^{-2t} - \tfrac{t}{2}e^{-2t} - \tfrac{t^2}{2}e^{-2t}$,

$\int te^{-2t}dt = -\tfrac{1}{4}e^{-2t} - \tfrac{t}{2}e^{-2t}$ and $\int e^{-2t}dt = -\tfrac{1}{2}e^{-2t}$

$\therefore$ By substituting (2) and (3) in (1), we get

$$e_X(x) = \frac{1+x+\frac{1}{2}x^2}{1+x+x^2},$$

$$E(X \wedge x) = -\int_{-\infty}^{0} F(t)dt + \int_{0}^{x} S(t)dt$$

$$\Rightarrow E(X \wedge x) = 0 + \int_0^x (1+t+t^2)e^{-2t}dt$$

$$= -(1+t+\tfrac{1}{2}t^2)e^{-2t}\Big|_0^x$$

$$= 1-(1+x+\tfrac{1}{2}x^2)e^{-2x}.$$

Q2: [4+4]

(a)

$\because X = \text{Pareto}(8/3, 8000)$

The K^{th} moment is given by $\text{E}(X^k) = \frac{\theta^k k!}{(\alpha-1)\ldots(\alpha-k)}$

$\therefore \text{E}(X) = \mu = \frac{\theta}{(\alpha-1)} = \frac{8{,}000}{\frac{8}{3}-1} = 4{,}800$ and $\text{E}(X^2) = \frac{\theta^2 2!}{(\alpha-1)(\alpha-2)} = 115{,}200{,}000$

$\therefore \text{Var}(X) = \text{E}(X^2) - \mu^2$

$= 115{,}200{,}000 - 4{,}800^2$

$= 92{,}160{,}000$

For independent random variables $X_1, X_1, \ldots, X_{100}$, the sum is

$S_{100} = X_1 + X_2 + \cdots + X_{100}$

$\therefore$ by using central limit theorem

$E(S_{100}) = 100(4800) = 480{,}000$ and

$Var(S_{100}) = 100(92{,}160{,}000) = 9{,}216{,}000{,}000$

The standard deviation for the sum S_{100} is $\sqrt{Var(S_{100})} = 96{,}000$

$$\therefore \Pr(S_{100} > 700{,}000) = 1 - \Phi\left(\frac{700{,}000 - 480{,}000}{96{,}000}\right) = 1 - \Phi(2.29) \cong 0.011$$

(b)

$$F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \text{ i.e. } X \sim Pareto\ (\alpha, \theta)$$

For $Y = \ln(1+X/\theta),\ F_Y(y) = \Pr(Y \le y)$

$$F_Y(y) = \Pr[\ln(1+X/\theta) \le y]$$

$$\therefore F_Y(y) = \Pr[1+X/\theta \le e^y]$$

$$= \Pr[X \le \theta(e^y - 1)]$$

$$= 1 - \left[\frac{\theta}{\theta(e^y-1)+\theta}\right]^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{e^y}\right)^{\alpha}$$
$$= 1 - e^{-\alpha y},$$

which is the exponential distribution function with parameter $1/\alpha$.

Q3: [3+3+3]

We have, $F_X(x) = 1 - e^{-x}$ (exp, dist. with no scale parameter), so we could obtain the following:

(1) The inverse exponential distribution with no scale parameter (where $\tau = -1$) has cdf

$F_Y(y) = 1 - F_X(y^{-1})$ Theorem

$= 1 - [1 - e^{-1/y}]$

$F_Y(y) = e^{-1/y}$

With the scale parameter added, it is $F(y) = e^{-\theta/y}$ (inverse exponential distribution)

(2) The transformed exponential distribution with no scale parameter (where $\tau > 0$) has cdf

$F_Y(y) = F_X(y^{\tau}),$ $\tau>0$ Theorem

$= 1 - e^{-y^{\tau}}$

$F_Y(y) = 1 - exp(-y^{\tau})$

With the scale parameter added, it is $F(y) = 1 - \exp[-(y/\theta)^{\tau}]$ (Weibull distribution)

(3) The inverse transformed exponential distribution with no scale parameter has cdf

$F_Y(y) = 1 - F_X(y^{-\tau})$ Theorem for negative τ

$= 1 - [1 - exp(-y^{-\tau})]$

$F_Y(y) = exp(-y^{-\tau})$

With the scale parameter added, it is

$F(y) = exp[-(y/\theta)^{-\tau}] = exp[-(\theta/y)^{\tau}]$ (inverse Weibull distribution)