King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S2-1445H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.
(Note that SND Table is attached in page 2)

Q1: [3+4+3]

(a) If the random variable X has probability density function $f(x) = 2e^{-2x}, x \geq 0$. Determine the survival, the hazard rate and the mean excess loss functions.
(b) The severities of individual claims have a Pareto distribution with parameters $\alpha = 8/3$ and $\theta = 8{,}000$. Use the central limit theorem to approximate the probability that the sum of 100 independent claims will exceed 700,000.
(c) Show that $\mathrm{E}(X) = e(d)S(d) + \mathrm{E}(X \wedge d)$

Q2: [3+3]

(a) Seventy-five percent of claims have a normal distribution with a mean of 4,000 and a variance of 1,000,000. The remaining 25% have a normal distribution with a mean of 3,000 and a variance of 1,000,000. Determine the probability that a randomly selected claim exceeds 5,000.
(b) Demonstrate that the Weibull distribution is a scale distribution.

Q3: [3+3+3]

Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed, and inverse transformed exponential distributions.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+4+3]

(a)

(i)

The survival function is

$$S(x) = 1 - F(x)$$

$$= \int_x^\infty 2e^{-2t} dt$$

$$= -e^{-2t} \Big|_x^\infty$$

$$\therefore S(x) = e^{-2x}$$

(ii)

$\because$ The hazard rate function is

$$h(x) = \frac{f(x)}{S(x)} = \frac{2e^{-2x}}{e^{-2x}} = 2$$

or

$$h(x) = -\frac{d}{dx}[\ln S(x)]$$

$$= -\frac{d}{dx}[\ln(e^{-2x})]$$

$$= -\frac{d}{dx}[-2x]$$

$$\therefore h(x) = 2$$

(iii)

The mean excess loss function is

$$e_X(d) = \frac{\int_d^\infty S(x)dx}{S(d)} \qquad (1)$$

From (i) $S(x) = e^{-2x}$ (2)

$$\therefore e_X(d) = \frac{\int_d^\infty e^{-2x}dx}{S(d)}$$

$$= \frac{\frac{1}{2}e^{-2d}}{e^{-2d}}$$

$$= \frac{1}{2} \qquad (3)$$

(b)

$\because X = \text{Pareto}(8/3, 8000)$

The K^{th} moment is given by $\text{E}(X^k) = \frac{\theta^k k!}{(\alpha-1)\ldots(\alpha-k)}$

$\therefore \text{E}(X) = \mu = \frac{\theta}{(\alpha-1)} = \frac{8{,}000}{\frac{8}{3}-1} = 4{,}800$ and $\text{E}(X^2) = \frac{\theta^2 2!}{(\alpha-1)(\alpha-2)} = 115{,}200{,}000$

$$\therefore \text{Var}(X) = \text{E}(X^2) - \mu^2$$

$$= 115{,}200{,}000 - 4{,}800^2$$

$$= 92{,}160{,}000$$

For independent random variables $X_1, X_1, \ldots, X_{100}$, the sum is

$S_{100} = X_1 + X_2 + \cdots + X_{100}$

$\therefore$ by using central limit theorem

$E(S_{100}) = 100(4800) = 480{,}000$ and

$Var(S_{100}) = 100(92{,}160{,}000) = 9{,}216{,}000{,}000$

The standard deviation for the sum S_{100} is $\sqrt{Var(S_{100})} = 96{,}000$

$\therefore \Pr(S_{100} > 700{,}000) = 1 - \Phi\left(\frac{700{,}000 - 480{,}000}{96{,}000}\right) = 1 - \Phi(2.29) \cong 0.011$

(c)

$\because \mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$

$= \int_{-\infty}^{d} xf(x)dx + \int_{d}^{\infty} (x-d)f(x)dx + d\int_{d}^{\infty} f(x)dx$

$\therefore \mathrm{E}(X) = \int_{-\infty}^{d} xf(x)dx + \int_{d}^{\infty} (x-d)f(x)dx + dS(d)$

$\because \mathrm{E}(X-d)_+ = \int_{d}^{\infty} (x-d)f(x)dx = e(d)S(d)$

and

$\because \mathrm{E}(X \wedge d) = \int_{-\infty}^{d} xf(x)dx + dS(d)$

$\therefore \mathrm{E}(X) = e(d)S(d) + \mathrm{E}(X \wedge d)$

Q2: [3+3]

(a)

For this mixture distribution,

$F(5000) = 0.75\Phi\left(\frac{5000-4000}{1000}\right) + 0.25\Phi\left(\frac{5000-3000}{1000}\right)$

$= 0.75\Phi(1) + 0.25\Phi(2)$

$= 0.75(0.8413) + 0.25(0.9772)$

$= 0.8753$

$\therefore Pr(X > 5000) = 1 - 0.8753 \cong 0.1247$, where X is a randomly selected claim.

(b)

For the Weibull distribution, $X \sim$ Weibull (θ, τ)

and the distribution function is $F_X(x) = 1 - e^{-(x/\theta)^\tau}$

let $Y = cX$, $c > 0$, then

$F_Y(y) = pr(Y \leq y)$

$= pr(X \leq \frac{y}{c})$

$\therefore F_Y(y) = 1 - e^{-(y/c\theta)^\tau}$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore \theta$ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution.

Q3: [3+3+3]

We have, $F_X(x) = 1 - e^{-x}$ (exp, dist. with no scale parameter), so we could obtain the following:

(1) The inverse exponential distribution with no scale parameter (where $\tau = -1$) has cdf

$F_Y(y) = 1 - F_X(y^{-1})$ Theorem

$= 1 - [1 - e^{-1/y}]$

$F_Y(y) = e^{-1/y}$

With the scale parameter added, it is $F(y) = e^{-\theta/y}$ (inverse exponential distribution)

(2) The transformed exponential distribution with no scale parameter (where $\tau > 0$) has cdf

$F_Y(y) = F_X(y^{\tau}),$ $\tau>0$

$= 1 - e^{-y^{\tau}}$

$F_Y(y) = 1 - exp(-y^{\tau})$

With the scale parameter added, it is $F(y) = 1 - \exp[-(y/\theta)^{\tau}]$ (Weibull distribution)

(3) The inverse transformed exponential distribution with no scale parameter has cdf

$F_Y(y) = 1 - F_X(y^{-\tau})$ Theorem for negative τ

$= 1 - [1 - exp(-y^{-\tau})]$

$F_Y(y) = exp(-y^{-\tau})$

With the scale parameter added, it is

$F(y) = exp[-(y/\theta)^{-\tau}] = exp[-(\theta/y)^{\tau}]$ (inverse Weibull distribution)