King Saud University
College of Sciences
Department of Mathematics

First Mid-Term Exam, S1-1448H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: [2+2+2+1+1]

Determine the mean excess loss, limited expected value and probability density functions for the following model.

$$F(x)=\begin{cases}0, & x<0\\ 1-\left(\frac{2{,}000}{x+2{,}000}\right)^3, & x\geq 0\end{cases}$$

and show that this model is a member of the transformed beta family, and give its name.

Note that: the pdf of generalized beta is defined as $f(x)=\frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$.

Q2: [4+4]

(a) One hundred observed claims in 2023 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 5% per year. Based on the empirical distribution from 2023, determine a range for the probability that a claim exceeds 500 in 2026.

(b) Demonstrate that the Weibull distribution is a scale distribution.
Hint: For Weibull distribution, $F(x)=1-e^{-(x/\theta)^{\tau}}$.

Q3: [3+3+3]

Let X have cdf $F_X(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Clarify the names of distributions.

The Model Answer

Q1: [2+2+2+1+1]

(i)

The mean excess loss function is

$$e_X(d) = \frac{\int_d^{\infty} S(x)dx}{S(d)}, \quad S(x) = \left(\frac{2000}{x+2000}\right)^3$$

$$\therefore e_X(d) = \frac{\int_d^{\infty} \left(\frac{2000}{x+2000}\right)^3 dx}{\left(\frac{2000}{d+2000}\right)^3}$$

$$= \frac{2000+d}{2}$$

(ii)

To get the limited expected value function $E(X \wedge u)$

$$E(X \wedge u) = -\int_{-\infty}^{0} F(x)dx + \int_0^u S(x)dx$$

$$\Rightarrow E(X \wedge u) = 0 + \int_0^u \left(\frac{2000}{x+2000}\right)^3 dx$$

$$= (2000)^3 \left[\frac{(x+2000)^{-2}}{-2}\right]_0^u$$

$$\therefore E(X \wedge u) = 1000\left[1 - \frac{4{,}000{,}000}{(u+2000)^2}\right]$$

(iii)

To get the pdf of the given model

$$f(x) = F'(x)$$

$$= \frac{3(2000)^3}{(x+2000)^4}, \quad x > 0$$

Which is the pdf of the Pareto distribution.

(iv)

For $X \sim$ Transformed beta $(\alpha, \theta, \gamma, \tau)$ generalized beta

$$f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad (1)$$

$at\ \gamma=\tau=1$

$$(1)\Rightarrow f(x)=\frac{\Gamma(\alpha+1)(x/\theta)}{\Gamma(\alpha)\Gamma(1)x[1+(x/\theta)]^{\alpha+1}}$$

$$=\frac{\alpha\Gamma(\alpha)(x/\theta)}{\Gamma(\alpha)x[1+(x/\theta)]^{\alpha+1}}$$

$$=\frac{\alpha}{\theta}/\left(\frac{x+\theta}{\theta}\right)^{\alpha+1}$$

$\therefore f(x)=\dfrac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$ which is a Pareto Prob. density function (2)

$\therefore$ The given model where $\alpha=3,\ \theta=2000$ is a member of the transformed beta family.

Q2: [4+4]

(a)

The claim amount in 2023	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 5% per year
In 2024 → 1.05 X, in 2025→ 1.1025 X and in 2026 → 1.157625 X
Where, X is the random variable of the claim amount in 2023 and Y=1.157625 X is the random variable of the claim amount in 2026. $Pr(Y>500)=Pr(X>500/1.157625)=Pr(X>431.9)$
From given data, $Pr(X>400)$ =50/100=0.5 and $Pr(X>450)$ =45/100=0.45

$\therefore$ 0.45<Pr($Y>500$) < 0.50

(b)

For the Weibull distribution, $X\sim$ Weibull (θ,τ)

and the distribution function is $F_X(x)=1-e^{-(x/\theta)^{\tau}}$

let $Y=cX,\ c>0$, then

$$F_Y(y)=pr(Y\le y)$$

$$=pr(X\le\tfrac{y}{c})$$

$i.e.\ F_Y(y)=F_X(\frac{y}{c})$

$\therefore\ F_Y(y)=1-e^{-(y/c\theta)^{\tau}}$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore\ \theta$ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution.

Q3: [3+3+3]

For Pareto (α,θ) distribution, $F_X(x)=1-\left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$.

So, we could obtain the following:
(1) The inverse distribution has cdf

$$F_Y(y)=1-F_X\left(y^{-1}\right),\quad \tau=-1$$

$$= \left(\frac{\theta}{y^{-1} + \theta} \right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{y}{y + \theta^{-1}} \right)^{\alpha}$$

which is the inverse Pareto (α, θ^{-1}) distribution.

(2) The transformed distribution has cdf

$$F_Y(y) = F_X(y^{\tau}), \qquad \tau > 0$$

$$= 1 - \left(\frac{\theta}{y^{\tau} + \theta} \right)^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1 + (y / \theta^{1/\tau})^{\tau}} \right)^{\alpha}$$

which is the Burr $(\alpha, \theta^{1/\tau}, \tau)$ distribution.

(3) The inverse transformed distribution has cdf

$$F_Y(y) = 1 - F_X\left(y^{-\tau}\right) \qquad \text{Theorem for negative } \tau$$

$$= 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}} \right)^{\alpha} \right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}} \right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}} \right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + (\theta^{-1/\tau})^{\tau}} \right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{(y / \theta^{-1/\tau})^{\tau}}{1 + (y / \theta^{-1/\tau})^{\tau}} \right)^{\alpha}$$

which is the inverse Burr $(\alpha, \theta^{-1/\tau}, \tau)$ distribution.
