King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S1-1446H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: [6+3+1]

An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses:
100(0.4), 500(0.2), 1000(0.2), 2500(0.1), and 10000(0.1). Determine the probability mass functions and expected values for each of the following.
(a) The excess loss and left censored and shifted variables, where the deductible is set at 750.
(b) The limited loss variable with a limit of 750.
(c) Show that the sum of the expected values of the limited loss and left censored and shifted random variables is equal to the expected value of the original random variable.

Q2: [3+3]

(a) Obtain the mgf and pgf for the Poisson distribution.
(b) Demonstrate that the Weibull distribution is a scale distribution.

Q3: [3+3+3]

Let X have cdf $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Clarify the names of distributions.

The Model Answer

Q1: [6+3+1]

(a)

x_j	100	500	1000	2500	10000
$p(x_j)$	0.4	0.2	0.2	0.1	0.1

$$S(d) = S(750)$$
$$= \Pr(X > 750)$$
$$= 0.4$$

For the mean of excess loss variable

$x_j - d$	250	1750	9250
$\dfrac{p(x_j)}{S(d)}$	0.5	0.25	0.25

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2875$$

For the mean of left censored and shifted variable

$$E(Y^L) = E\left[(X-d)_+\right]$$

$x_j - d$	0	250	1750	9250
$p(x_j)$	0.6	0.2	0.1	0.1

$$\therefore E(Y^L) = 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$$
$$= 1150$$

(b)

For the mean of limited loss variable

$x_j \wedge d$	100	500	750
$p(x_j)$	0.4	0.2	0.4

$$E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$$
$$= 440$$

(c)

$$E(X) = 100(0.4) + 500(0.2) + 1000(0.2) + 2500(0.1) + 10000(0.1)$$
$$= 1590$$

Clearly, $E(X) = E\left[(X-d)_+\right] + E(X \wedge d)$

Q2: [3+3]

(a)

The pgf is

$$P_X(z) = \sum_{x=0}^{\infty} z^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(z\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{z\lambda}$$

$$\therefore P_X(z) = e^{\lambda(z-1)}$$

The mgf is

$$M_X(z) = P_X(e^z)$$

$$\therefore M_X(z) = \exp[\lambda(e^z - 1)]$$

(b)

For the Weibull distribution, $X \sim$ Weibull (θ, τ)

and the distribution function is $F_X(x) = 1 - e^{-(x/\theta)^\tau}$

let $Y = cX$, $c > 0$, then

$$F_Y(y) = pr(Y \le y)$$

$$= pr(X \le \tfrac{y}{c})$$

$$\therefore \ F_Y(y) = 1 - e^{-(y/c\theta)^\tau}$$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore$ θ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution

Q3: [3+3+3]

For Pareto (α, θ) distribution, $F_X(x) = 1 - \left(\dfrac{\theta}{x+\theta}\right)^\alpha$.

So, we could obtain the following:

(1) The inverse distribution has cdf

$$F_Y(y) = 1 - F_X(y^{-1}), \quad \tau = -1$$

$$= \left(\frac{\theta}{y^{-1}+\theta}\right)^\alpha$$

$$\therefore F_Y(y) = \left(\frac{y}{y+\theta^{-1}}\right)^\alpha$$

which is the inverse Pareto (α, θ^{-1}) distribution.

(2) The transformed distribution has cdf

$$F_Y(y) = F_X(y^\tau), \qquad \tau > 0$$

$$= 1 - \left(\frac{\theta}{y^\tau + \theta}\right)^\alpha$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1 + (y/\theta^{1/\tau})^{\tau}} \right)^{\alpha}$$

which is the Burr $(\alpha, \theta^{1/\tau}, \tau)$ distribution.

(3) The inverse transformed distribution has cdf

$$F_Y(y) = 1 - F_X(y^{-\tau}) \qquad \text{Theorem for negative } \tau$$

$$= 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}} \right)^{\alpha} \right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}} \right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}} \right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + (\theta^{-1/\tau})^{\tau}} \right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{(y/\theta^{-1/\tau})^{\tau}}{1 + (y/\theta^{-1/\tau})^{\tau}} \right)^{\alpha}$$

which is the inverse Burr $(\alpha, \theta^{-1/\tau}, \tau)$ distribution.
