King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S1-1445H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: [3+3+1.5+1.5]

Determine the mean excess loss, limited expected value and probability density functions for the following model.

$$F(x)=\begin{cases}0, & x<0\\ 1-\left(\dfrac{2{,}000}{x+2{,}000}\right)^3, & x\geq 0\end{cases}$$

and show that this model is a member of the transformed beta family.

Note that: the pdf of generalized beta is defined as $f(x)=\dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$

Q2: [3+2+2]

(a) Obtain the mgf and pgf for the Poisson distribution.
(b) Demonstrate that the Weibull distribution is a scale distribution.
(c) Write down the probability density function of a 50-50 mixture of two gamma distributions. One has parameters $\alpha=4$ and $\theta=7$ and the other has parameters $\alpha=15$ and $\theta=7$.

Q3: [3+3+3]

Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed, and inverse transformed exponential distributions.

The Model Answer

Q1: [3+3+1.5+1.5]

(i)

The mean excess loss function is

$$e_X(d)=\frac{\int_d^\infty S(x)dx}{S(d)}, \quad S(x)=\left(\tfrac{2000}{x+2000}\right)^3$$

$$\therefore e_X(d)=\frac{\int_d^\infty \left(\tfrac{2000}{x+2000}\right)^3 dx}{\left(\tfrac{2000}{d+2000}\right)^3}$$

$$=\tfrac{2000+d}{2}$$

(ii)

To get the limited expected value function $E(X\wedge u)$

$$E(X\wedge u)=-\int_{-\infty}^{0}F(x)dx+\int_0^u S(x)dx$$

$$\Rightarrow\ E(X\wedge u)=0+\int_0^u\left(\tfrac{2000}{x+2000}\right)^3dx$$

$$=(2000)^3\left[\frac{(x+2000)^{-2}}{-2}\right]_0^u$$

$$\therefore\ E(X\wedge u)\ =1000\left[1-\frac{4{,}000{,}000}{(u+2000)^2}\right]$$

(iii)

To get the pdf of the given model

$$f(x)=F'(x)$$

$$=\tfrac{3(2000)^3}{(x+2000)^4},\quad x>0$$

Which is the pdf of the Pareto distribution.

(iv)

For X ~ Transformed beta $(\alpha,\theta,\gamma,\tau)$ generalized beta

$$f(x)=\frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad (1)$$

$at\ \gamma = \tau = 1$

$$(1) \Rightarrow f(x) = \frac{\Gamma(\alpha+1)(x/\theta)}{\Gamma(\alpha)\Gamma(1)x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha!(x/\theta)}{(\alpha-1)!x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha}{\theta} / \left(\frac{x+\theta}{\theta}\right)^{\alpha+1}$$

$\therefore f(x) = \dfrac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$ which is a Pareto Prob. density function (2)

$\therefore$ The given model is a member of the transformed beta family.

Q2: [3+2+2]

(a)

The pgf is

$$P_X(z) = \sum_{x=0}^{\infty} z^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(z\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{z\lambda}$$

$\therefore P_X(z) = e^{\lambda(z-1)}$

The mgf is

$M_X(z) = P_X(e^z)$

$\therefore M_X(z) = \exp[\lambda(e^z - 1)]$

(b)

For the Weibull distribution, $X \sim$ Weibull (θ, τ)

and the distribution function is $F_X(x) = 1 - e^{-(x/\theta)^{\tau}}$

let $Y = cX,\ c > 0$, then

$$F_Y(y) = pr(Y \le y)$$

$$= pr(X \le \tfrac{y}{c})$$

$\therefore\ F_Y(y) = 1 - e^{-(y/c\theta)^{\tau}}$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore\ \theta$ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution

(c)

$$f(x) = 0.5\frac{(x/7)^4 e^{-x/7}}{x\Gamma(4)} + 0.5\frac{(x/7)^{15} e^{-x/7}}{x\Gamma(15)}$$

$$\therefore f(x) = 0.5\frac{x^3 e^{-x/7}}{3!7^4} + 0.5\frac{x^{14} e^{-x/7}}{14!7^{15}}$$

Q3: [3+3+3]

We have, $F_X(x) = 1 - e^{-x}$ (exp, dist. with no scale parameter), so we could obtain the following:

(1) The inverse exponential distribution with no scale parameter (where $\tau = -1$) has cdf

$$F_Y(y) = 1 - F_X(y^{-1}) \qquad \text{Theorem}$$

$$= 1 - [1 - e^{-1/y}]$$

$$F_Y(y) = e^{-1/y}$$

With the scale parameter added, it is $F(y) = e^{-\theta/y}$ (inverse exponential distribution)

(2) The transformed exponential distribution with no scale parameter (where $\tau > 0$) has cdf

$$F_Y(y) = F_X(y^\tau), \qquad \tau > 0$$

$$= 1 - e^{-y^\tau}$$

$$F_Y(y) = 1 - \exp(-y^\tau)$$

With the scale parameter added, it is $F(y) = 1 - \exp[-(y/\theta)^\tau]$ (Weibull distribution)

(3) The inverse transformed exponential distribution with no scale parameter has cdf

$$F_Y(y) = 1 - F_X(y^{-\tau}) \qquad \text{Theorem for negative } \tau$$

$$= 1 - [1 - \exp(-y^{-\tau})]$$

$$F_Y(y) = \exp(-y^{-\tau})$$

With the scale parameter added, it is

$F(y) = \exp[-(y/\theta)^{-\tau}] = \exp[-(\theta/y)^\tau]$ (inverse Weibull distribution)