



Answer the following questions.

Q1: 10[2+6+2]

- (a) If the random variable X has the survival function $S(x) = (1 + x + x^2)e^{-2x}$, $x \geq 0$, Determine the probability density and hazard rate functions.
- (b) An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses:
100(0.4), 500(0.2), 1,000(0.2), 2,500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for each of the following.
- (i) The excess loss and left censored and shifted variables, where the deductible is set at 750.
- (ii) The limited loss variable with a limit of 750.
- (c) Show that $E(X) = e(d)S(d) + E(X \wedge d)$

Q2: [3+3]

- (a) One hundred observed claims in 2022 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 5% per year. Based on the empirical distribution from 2022, determine a range for the probability that a claim exceeds 500 in 2025.
- (b) Let $X|\Lambda$ have an exponential distribution with parameter $1/\Lambda$. Let Λ have a gamma distribution. Determine the unconditional (mixture) distribution of X .
Hint: replace θ in the gamma distribution by its reciprocal.

Q3: [3+3+3]

Let X have cdf $F_X(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^\alpha$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Clarify the names of distributions.

The Model Answer

Q1: 10[2+6+2]

(a)

$$\because S(x) = (1+x+x^2)e^{-2x}, \quad x \geq 0$$

The probability density function is

$$\begin{aligned} f(x) &= -S'(x) \\ &= -(1+2x)e^{-2x} + 2(1+x+x^2)e^{-2x} \end{aligned}$$

$$\therefore f(x) = (1+2x^2)e^{-2x}$$

The hazard rate function is

$$h(x) = -\frac{d}{dx} [\ln S(x)]$$

$$\because S(x) = (1+x+x^2)e^{-2x},$$

$$\ln S(x) = -2x + \ln(1+x+x^2)$$

$$\therefore h(x) = 2 - \frac{1+2x}{1+x+x^2}$$

or simply,

$$h(x) = \frac{f(x)}{S(x)} = \frac{1+2x^2}{1+x+x^2}$$

(b)

(i)

x_j	100	500	1,000	2,500	10,000
$p(x_j)$	0.4	0.2	0.2	0.1	0.1

$$\begin{aligned} S(d) &= S(750) \\ &= \Pr(X > 750) \\ &= 0.4 \end{aligned}$$

For the mean of excess loss variable

$x_j - d$	250	1,750	9,250
$\frac{p(x_j)}{S(d)}$	0.5	0.25	0.25

$$\begin{aligned} \therefore e_x(d) &= 250(0.5) + 1750(0.25) + 9250(0.25) \\ &= 2875 \end{aligned}$$

For the mean of left censored and shifted variable

$$E(Y^L) = E[(X - d)_+]$$

$x_j - d$	0	250	1,750	9,250
$p(x_j)$	0.6	0.2	0.1	0.1

$$\begin{aligned} \therefore E(Y^L) &= 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1) \\ &= 1150 \end{aligned}$$

(ii)

For the mean of limited loss variable

$x_j \wedge d$	100	500	750
$p(x_j)$	0.4	0.2	0.4

$$\begin{aligned} E(X \wedge d) &= 100(0.4) + 500(0.2) + 750(0.4) \\ &= 440 \end{aligned}$$

(c)

$$\begin{aligned} \therefore E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_{-\infty}^d xf(x)dx + \int_d^{\infty} (x - d)f(x)dx + d \int_d^{\infty} f(x)dx \\ \therefore E(X) &= \int_{-\infty}^d xf(x)dx + \int_d^{\infty} (x - d)f(x)dx + dS(d) \end{aligned}$$

$$\therefore E(X - d)_+ = \int_d^{\infty} (x - d)f(x)dx = e(d)S(d)$$

and

$$\therefore E(X \wedge d) = \int_{-\infty}^d xf(x)dx + dS(d)$$

$$\therefore E(X) = e(d)S(d) + E(X \wedge d)$$

Q2: [3+3]

(a)

The claim amount in 2022	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 5% per year

In 2023 $\rightarrow 1.05 X$, in 2024 $\rightarrow 1.1025 X$ and in 2025 $\rightarrow 1.157625 X$

Where, X is the random variable of the claim amount in 2022 and $Y = 1.157625 X$ is the random variable of the claim amount in 2025. $Pr(Y > 500) = Pr(X > 500/1.157625) = Pr(X > 431.9)$

From given data, $Pr(X > 400) = 50/100 = 0.5$ and $Pr(X > 450) = 45/100 = 0.45$

$$\therefore 0.45 < Pr(Y > 500) < 0.50$$

(b)

Let $X|\Lambda \sim \exp(\lambda)$, $\Lambda \sim \text{gamma}(\alpha, \theta)$

$$\therefore f_X(x) = \int f_{X|\Lambda}(x|\lambda)f_{\Lambda}(\lambda)d\lambda$$

$$\begin{aligned}
f_X(x) &= \frac{\theta^\alpha}{\Gamma(\alpha)} \int \lambda e^{-\lambda x} \lambda^{\alpha-1} e^{-\theta \lambda} d\lambda \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)} d\lambda \\
&= \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} du \\
&= \frac{\theta^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} = \frac{\theta^\alpha \alpha \Gamma(\alpha)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}}
\end{aligned}$$

$$\therefore f_X(x) = \frac{\alpha \theta^\alpha}{(x+\theta)^{\alpha+1}},$$

Which is the pdf of the Pareto (α, θ) distribution.

Q3: [3+3+3]

For Pareto (α, θ) distribution, $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$.

So, we could obtain the following:

(1) The inverse distribution has cdf

$$\begin{aligned}
F_Y(y) &= 1 - F_X(y^{-1}), \quad \tau = -1 \\
&= \left(\frac{\theta}{y^{-1} + \theta}\right)^\alpha
\end{aligned}$$

$$\therefore F_Y(y) = \left(\frac{y}{y + \theta^{-1}}\right)^\alpha$$

which is the inverse Pareto (α, θ^{-1}) distribution.

(2) The transformed distribution has cdf

$$\begin{aligned}
F_Y(y) &= F_X(y^\tau), \quad \tau > 0 \\
&= 1 - \left(\frac{\theta}{y^\tau + \theta}\right)^\alpha
\end{aligned}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1 + (y/\theta^{1/\tau})^\tau}\right)^\alpha$$

which is the Burr $(\alpha, \theta^{1/\tau}, \tau)$ distribution.

(3) The inverse transformed distribution has cdf

$$F_Y(y) = 1 - F_X(y^{-\tau}) \quad \text{Theorem for negative } \tau$$

$$= 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}} \right)^\alpha \right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}} \right)^\alpha$$

$$= \left(\frac{y^\tau}{y^\tau + \theta^{-1}} \right)^\alpha$$

$$= \left(\frac{y^\tau}{y^\tau + (\theta^{-1/\tau})^\tau} \right)^\alpha$$

$$\therefore F_Y(y) = \left(\frac{(y / \theta^{-1/\tau})^\tau}{1 + (y / \theta^{-1/\tau})^\tau} \right)^\alpha$$

which is the inverse Burr $(\alpha, \theta^{-1/\tau}, \tau)$ distribution.
