King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S1-1447H
ACTU 475
Credibility Theory and Loss Distributions
Time: 90 Minutes - Marks: 25

Answer the following questions.

Q1: 10[2+6+2]

(a) If the random variable X has the survival function $S(x)=(1+x+x^2)e^{-2x},\ x\geq 0$, Determine the probability density and hazard rate functions.

(b) An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses:
100(0.4), 500(0.2), 1,000(0.2), 2,500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for each of the following.

(i) The excess loss and left censored and shifted variables, where the deductible is set at 750.

(ii) The limited loss variable with a limit of 750.

(c) Show that $\mathrm{E}(X)=e(d)S(d)+\mathrm{E}(X\wedge d)$

Q2: [3+3]

(a) One hundred observed claims in 2022 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 5% per year. Based on the empirical distribution from 2022, determine a range for the probability that a claim exceeds 500 in 2025.

(b) Let $X|\Lambda$ have an exponential distribution with parameter $1/\Lambda$. Let Λ have a gamma distribution. Determine the unconditional (mixture) distribution of X.

Hint: replace θ in the gamma distribution by its reciprocal.

Q3: [3+3+3]

Let X have cdf $F_X(x)=1-\left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Clarify the names of distributions.

The Model Answer

Q1: 10[2+6+2]

(a)

$$\because S(x) = (1+x+x^2)e^{-2x},\ x \ge 0$$

The probability density function is

$$f(x) = -S'(x)$$
$$= -(1+2x)e^{-2x} + 2(1+x+x^2)e^{-2x}$$
$$\therefore f(x) = (1+2x^2)e^{-2x}$$

The hazard rate function is

$$h(x) = -\frac{d}{dx}[\ln S(x)]$$
$$\because S(x) = (1+x+x^2)e^{-2x},$$
$$\ln S(x) = -2x + \ln(1+x+x^2)$$
$$\therefore\ h(x) = 2 - \frac{1+2x}{1+x+x^2}$$

or simply,

$$h(x) = \frac{f(x)}{S(x)} = \frac{1+2x^2}{1+x+x^2}$$

(b)

(i)

x_j	100	500	1,000	2,500	10,000
$p(x_j)$	0.4	0.2	0.2	0.1	0.1

$$S(d) = S(750)$$
$$= \Pr(X > 750)$$
$$= 0.4$$

For the mean of excess loss variable

$x_j - d$	250	1,750	9,250
$\dfrac{p(x_j)}{S(d)}$	0.5	0.25	0.25

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2875$$

For the mean of left censored and shifted variable

$$E(Y^L) = E\left[(X-d)_+\right]$$

$x_j - d$	0	250	1,750	9,250
$p(x_j)$	0.6	0.2	0.1	0.1

$$\therefore E(Y^L) = 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$$
$$= 1150$$

(ii)

For the mean of limited loss variable

$x_j \wedge d$	100	500	750
$p(x_j)$	0.4	0.2	0.4

$$E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$$
$$= 440$$

(c)

$$\because \mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx$$
$$= \int_{-\infty}^{d} xf(x)dx + \int_{d}^{\infty} (x-d)f(x)dx + d\int_{d}^{\infty} f(x)dx$$
$$\therefore \mathrm{E}(X) = \int_{-\infty}^{d} xf(x)dx + \int_{d}^{\infty} (x-d)f(x)dx + dS(d)$$

$$\because \mathrm{E}(X-d)_+ = \int_{d}^{\infty} (x-d)f(x)dx = e(d)S(d)$$

and

$$\because \mathrm{E}(X \wedge d) = \int_{-\infty}^{d} xf(x)dx + dS(d)$$
$$\therefore \mathrm{E}(X) = e(d)S(d) + \mathrm{E}(X \wedge d)$$

Q2: [3+3]

(a)

The claim amount in 2022		0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims		42	3	5	5	0	5	40

For the next three years, all claims are inflated by 5% per year

In 2023 → 1.05 X, in 2024→ 1.1025 X and in 2025 → 1.157625 X

Where, X is the random variable of the claim amount in 2022 and Y=1.157625 X is the random variable of the claim amount in 2023. $Pr(Y > 500) = Pr(X > 500/1.157625) = Pr(X > 431.9)$

From given data, $Pr(X > 400)$ =50/100=0.5 and $Pr(X > 450)$ =45/100=0.45

$\therefore$ 0.45<$\Pr(Y > 500) < 0.50$

(b)

Let $X|\Lambda \sim \exp(\lambda),\ \Lambda \sim \mathrm{gamma}(\alpha, \theta)$

$$\because f_X(x) = \int f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda) d\lambda$$

$$f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} \int \lambda e^{-\lambda x} \lambda^{\alpha-1} e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^\alpha e^{-\lambda(x+\theta)} d\lambda$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} \int_0^\infty u^\alpha e^{-u} du$$

$$= \frac{\theta^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}} = \frac{\theta^\alpha \alpha \Gamma(\alpha)}{\Gamma(\alpha)(x+\theta)^{\alpha+1}}$$

$$\therefore f_X(x) = \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}},$$

Which is the pdf of the Pareto (α,θ) distribution.

Q3: [3+3+3]

For Pareto (α,θ) distribution, $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$.

So, we could obtain the following:

(1) The inverse distribution has cdf

$$F_Y(y) = 1 - F_X(y^{-1}), \quad \tau = -1$$

$$= \left(\frac{\theta}{y^{-1}+\theta}\right)^\alpha$$

$$\therefore F_Y(y) = \left(\frac{y}{y+\theta^{-1}}\right)^\alpha$$

which is the inverse Pareto (α,θ^{-1}) distribution.

(2) The transformed distribution has cdf

$$F_Y(y) = F_X(y^\tau), \qquad \tau > 0$$

$$= 1 - \left(\frac{\theta}{y^\tau + \theta}\right)^\alpha$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1+(y/\theta^{1/\tau})^\tau}\right)^\alpha$$

which is the Burr $(\alpha,\theta^{1/\tau},\tau)$ distribution.

(3) The inverse transformed distribution has cdf

$$F_Y(y) = 1 - F_X\left(y^{-\tau}\right) \qquad \text{Theorem for negative } \tau$$

$$= 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + (\theta^{-1/\tau})^{\tau}}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{(y/\theta^{-1/\tau})^{\tau}}{1 + (y/\theta^{-1/\tau})^{\tau}}\right)^{\alpha}$$

which is the inverse Burr $(\alpha, \theta^{-1/\tau}, \tau)$ distribution.