King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2-1445H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions:

(Note that SND Table is attached in page 3)

Q1: [3+4]

(a) For the model of automobile bodily injury claim that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-\left(\dfrac{2000}{x+2000}\right)^3, & x\geq 0.\end{cases}$$

Determine the survival, density, and hazard rate functions.

(b) The cdf of a random variable X is $F(x)=1-\exp\left(-\dfrac{x}{\theta}\right),\ x>0.$

Find $e_X(x)$ and $E(X\wedge x)$.

Q2: [4+3]

(a) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and variance of the gamma distribution.

(b) Let X have cdf $F_X(x)=\dfrac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$. Determine the inverse distribution of X, with clarifying the names of distributions.

Q3: [5+5]

(a) An insurance company has decided to establish its full-credibility requirements for an individual state rate filing. The full-credibility standard is to be set so that the observed total amount of claims underlying the rate filing would be within 5% of the true value with probability 0.90. The claim frequency follows a Poisson distribution and the severity distribution has pdf

$$f(x)=\frac{100-x}{5{,}000},\quad 0\leq x\leq 100.$$

Determine the expected number of claims necessary to obtain full credibility using the normal approximation.

(b) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table

Year	1	2	3
Claims	475	550	400

Determine the full credibility and partial credibility through premium by assuming the normal approximation. Use $r = 0.05$ and $p = 0.90$.

Q4: [5+5]

(a) There are two types of drivers. Good drivers make up 75% of the population and in one year have zero claims with probability 0.8, one claim with probability 0.1, and two claims with probability 0.1. Bad drivers make up the other 25% of the population and have zero, one, or two claims with probabilities 0.6, 0.2, and 0.2, respectively.

(i) Describe this process by using the concept of the risk parameter Θ.

(ii) For a particular policyholder, suppose that we have observed $x_1 = 0$ and $x_2 = 1$ for past claims. Determine the posterior distribution of $\Theta | X_1 = 0,\ X_2 = 1$ and the predictive distribution of $X_3 | X_1 = 0,\ X_2 = 1$.

(b) Claim sizes have an exponential distribution with mean θ. For 80% of risks, $\theta = 8$, and for 20% of risks, $\theta = 2$. A randomly selected policy had a claim of size 5 in year 1. Determine both the Bayesian and Bühlmann estimates of the expected claim size in year 2.

Q5: [6]

Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimates of the parameter θ for the inverse exponential and inverse gamma with $\alpha = 2$ distributions. Also, find the value of the log-likelihood function in each case. Compare your estimates with the method of moments estimates.

Hint: For inverse gamma distribution $f(x;\theta) = \dfrac{(\theta / x)^{\alpha} e^{-\theta/x}}{x\,\Gamma(\alpha)}$ and $\mathrm{E}(X^k) = \dfrac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)},\ k < \alpha.$

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+4]

(a) The survival function is

$$S(x) = 1 - F(x)$$

$$\therefore\ S(x) = \left(\frac{2000}{x+2000}\right)^3,\ x \geq 0$$

The density function is

$$f(x) = F^{'}(x) = -S^{'}(x)$$

$$\therefore\ f(x)\ = \frac{3(2000)^3}{(x+2000)^4},\ x > 0$$

The hazard rate function

$$h(x) = \frac{f(x)}{S(x)}$$

$$\therefore\ h(x) = \frac{3}{(x+2000)},\ x > 0$$

(b) The mean excess loss function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)}$$

$$\because S(x) = \exp\left(-\frac{x}{\theta}\right)$$

$$\Rightarrow e_X(x) = \frac{\int_x^\infty \exp\left(-\frac{t}{\theta}\right)dt}{\exp\left(-\frac{x}{\theta}\right)}$$

$$\therefore e_X(x) = \frac{-\theta.\exp\left(-\frac{t}{\theta}\right)\Big|_x^\infty}{\exp\left(-\frac{x}{\theta}\right)} = \theta$$

$$\because E(X \wedge x) = E(X) - e(x)S(x)$$

$$\therefore E(X \wedge x) = \theta - \theta.\exp\left(-\frac{x}{\theta}\right)$$
$$= \theta(1 - e^{-x/\theta})$$

Q2: [4+3]

(a)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1} / \Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^{\alpha}},$$

where $r(\theta) = -1/\theta$, $\mathrm{q}(\theta) = \theta^{\alpha}$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean,

$$E(X) = \mu(\theta) = \frac{q^{'}(\theta)}{r^{'}(\theta)q(\theta)}$$
$$= \frac{\alpha\theta^{\alpha-1}}{1/\theta^2.\ \theta^{\alpha}} = \alpha\theta$$

and the variance,

$$Var(X) = v(\theta)$$
$$= \frac{\mu^{'}(\theta)}{r^{'}(\theta)}$$
$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

(b)

For loglogistic distribution with 2 parameters γ and θ, $F_X(x) = \dfrac{(x/\theta)^{\gamma}}{1+(x/\theta)^{\gamma}}$

For $Y = X^{-1}$, we have $F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \frac{(y^{-1}/\theta)^\gamma}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{1}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{(y\theta)^\gamma}{1+(y\theta)^\gamma}$$

Which is also loglogistic distribution with 2 parameters γ and θ, γ unchanged and $\theta \to 1/\theta$.

Q3: [5+5]

(a)

at $p = 0.90$, $\Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p/r)^2 = (1.645/0.05)^2 = 1082.41$

$$E(X) = \int_0^{100} x\left(\frac{100-x}{5000}\right)dx$$
$$= \int_0^{100} \frac{100x - x^2}{5,000}\,dx$$
$$= \frac{1}{5,000}\left[100(x^2/2) - x^3/3\right]_0^{100}$$
$$\therefore E(X) = \frac{100^3}{5,000}\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{100}{3}$$
$$E(X^2) = \int_0^{100} x^2\left(\frac{100-x}{5,000}\right)dx$$
$$= \int_0^{100} \frac{100x^2 - x^3}{5,000}\,dx = \frac{5,000}{3}$$
$$\therefore Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{5,000}{3} - \frac{10,000}{9} = \frac{5,000}{9}$$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = \dfrac{5,000}{9}$, $\theta^2 = \dfrac{10,000}{9}$

$\therefore$ The expected # of claims $= 1082.41[1+0.5]$

$= 1623.615$

(b)

As in part (a) $\lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

The mean is $\xi = E(X_j) = \dfrac{475+550+400}{3} = 475,$

The variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)^2}{n-1} = \dfrac{0^2 + 75^2 + 75^2}{2} = 5625$

For full credibility $n \geq \lambda_0 \left(\dfrac{\sigma}{\xi}\right)^2$

$\therefore n \geq 1082.41\left(\dfrac{5625}{475^2}\right)$

$\therefore n \geq 26.98529086$

The credibility factor is $Z = \sqrt{\frac{n}{\lambda_0 \sigma^2/\xi^2}}$

$= \sqrt{\frac{3}{26.98529086}} = 0.333424$

The partial credibility through premium is

$P_c = Z\overline{X} + (1-Z)M$

$= 0.333424(475) + (1-0.333424)(600)$

$\therefore P_c = 558.32$

Q4: [5+5]

(a) (i)

x	$\Pr(X = x \mid \Theta = G)$	$\Pr(X = x \mid \Theta = B)$	θ	$\Pr(\Theta = \theta)$
0	0.8	0.6	G	0.75
1	0.1	0.2	B	0.25
2	0.1	0.2		

(ii)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G|0,1) = \frac{f(0|G)f(1|G)\pi(G)}{f_X(0,1)}$$

where $f_X(0,1) = \sum_{\theta} f_{X_1|\Theta}(0|\theta) f_{X_2|\Theta}(1|\theta)\pi(\theta)$

$$\begin{aligned} f_X(0,1) &= 0.8(0.1)(0.75) + 0.6(0.2)(0.25) \\ &= 0.09 \end{aligned}$$

$$\pi(G|0,1) = \frac{0.8(0.1)(0.75)}{0.09} \simeq 0.67$$

$$\pi(B|0,1) = \frac{0.6(0.2)(0.25)}{0.09} \simeq 0.33$$

For the predictive distribution, the predictive probabilities are given by

$$\begin{aligned} f_{X_3|X}(0|0,1) &= \sum_{\theta} f(0|\theta)\pi(\theta|0,1) \\ &= f(0|G)\pi(G|0,1) + f(0|B)\pi(B|0,1) \\ &= 0.8(0.67) + 0.6(0.33) \\ &= 0.734, \end{aligned}$$

$$\begin{aligned} f_{X_3|X}(1|0,1) &= \sum_{\theta} f(1|\theta)\pi(\theta|0,1) \\ &= f(1|G)\pi(G|0,1) + f(1|B)\pi(B|0,1) \\ &= 0.1(0.67) + 0.2(0.33) \\ &= 0.133, \end{aligned}$$

and $f_{X_3|X}(2|0,1) = \sum_{\theta} f(2|\theta)\pi(\theta|0,1)$

$$\begin{aligned} &= f(2|G)\pi(G|0,1) + f(2|B)\pi(B|0,1) \\ &= 0.1(0.67) + 0.2(0.33) \\ &= 0.133. \end{aligned}$$

(b)

The Bayesian estimate of the expected claim size in year 2.

We have $\pi(\Theta = 8) = 0.80$ and $\pi(\Theta = 2) = 0.20$, and # of claims (claim size) is 5 in year 1.

$$E(X_2|X_1=5)=E(\Theta|X_1=5)$$
$$=\mu(\Theta{=}8)\pi(\Theta=8|X_1=5)+\mu(\Theta{=}2)\pi(\Theta=2|X_1=5)$$
$$\pi(\Theta=8|X_1=5)=\frac{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)}{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)+\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}$$
$$=\frac{(1/8)e^{-5/8}(0.8)}{(1/8)e^{-5/8}(0.8)+(1/2)e^{-5/2}(0.2)}=0.867035$$
Similarly, $\pi(\Theta=2|X_1=5)=\dfrac{\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}{\Pr(X_1=5|\Theta=8)\pi(\Theta=8)+\Pr(X_1=5|\Theta=2)\pi(\Theta=2)}$
$$=\frac{(1/2)e^{-5/2}(0.2)}{(1/8)e^{-5/8}(0.8)+(1/2)e^{-5/2}(0.2)}=0.132965$$
$$\therefore E(X_2|X_1=5)=8\times 0.867035+2\times 0.132965$$
$$=7.2022$$
The Bühlmann estimate of the expected claim size in year 2. To determine the Bühlmann credibility estimate, we should find the following quantities.

$$\mu=E[\mu(\Theta)]$$
$$=8(0.80)+(2)(0.20)=6.8,$$

$$a=\mathrm{var}[\mu(\Theta)]$$
$$=8^2(0.8)+2^2(0.2)-6.8^2=5.76,$$

$$v=E[v(\Theta)]$$
$$=\sum_\theta v(\theta)\pi(\theta)$$
$$=8^2\times 0.8+2^2\times 0.2=52,$$

Note that for $X\sim\exp(\theta)$ the mean $=E(X)=\theta$ and $\mathrm{var}(X)\ =\theta^2$

$$k=\frac{v}{a}$$
$$=\frac{52}{5.76}=9.02778,$$

$$Z=\frac{n}{n+k}$$
$$=\frac{1}{1+9.02778}$$
$$\therefore\ Z\ =0.099723.$$

The Bühlmann estimate is

$$\begin{aligned} E(X_2|100) = P_c &= Z\overline{X} + (1-Z)\mu \\ &= 0.099723\times 5 + (1-0.099723)\times 6.8 \\ &= 6.6205. \end{aligned}$$

Q5: [6]

(1) For inv. exponential distribution, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{x_j^2}$$

$$l(\theta) = \sum_{j=1}^{n} \ln\theta - \sum_{j=1}^{n} \theta/x_j - 2\sum_{j=1}^{n} \ln x_j$$

$$l(\theta) = n\ln\theta - \theta\sum_{j=1}^{n} x_j^{-1} - 2\sum_{j=1}^{n} \ln x_j$$

$\therefore\ l(\theta) = n\ln\theta - ny\theta - 2\sum_{j=1}^{n} \ln x_j$, where $y = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

To get max. estimate of θ (i.e. $\hat{\theta}$), set $l'(\theta) = 0$

$$l'(\theta) = n\theta^{-1} - ny = 0$$

$$\Rightarrow\ \theta^{-1} = y \ \text{ i.e. } \hat{\theta} = 1/y$$

$$\therefore\ \hat{\theta} \approx 84.7$$

(2) The value of the log-likelihood function for inv. exponential distribution is given by

$$\begin{aligned} l(\hat{\theta}) &= 7\ln\hat{\theta} - 7 - 2\sum_{j=1}^{7} \ln x_j \\ &= 7\ln(84.702347) - 7 - 2(32.90166107) \\ &\approx -41.73 \end{aligned}$$

(3) For inv. gamma distribution with $\alpha = 2$, the likelihood function is

$$L(\theta)=\prod_{j=1}^{n}\theta^2 x_j^{-3} e^{-\theta/x_j}$$

$$l(\theta)=2\sum_{j=1}^{n}\ln\theta-3\sum_{j=1}^{n}\ln x_j-ny\theta,$$ where $y=\frac{1}{n}\sum_{j=1}^{n}x_j^{-1}$

$$l(\theta)=2n\ln\theta-ny\theta-3\sum_{j=1}^{n}\ln x_j$$

To get $\hat{\theta}$, let $l'(\theta)=0$

$$l'(\theta)=2n\theta^{-1}-ny=0$$

$$\Rightarrow\ \theta^{-1}=y/2\ \text{ i.e. } \hat{\theta}=2/y$$

$$\therefore\ \hat{\theta}\approx 169.4$$

(4) The value of the log-likelihood function for inv. gamma distribution is given by

$$l(\hat{\theta})=14\ln\hat{\theta}-14-3\sum_{j=1}^{7}\ln x_j$$

$$=14\ln(169.404694)-14-3(32.90166107)$$

$$\approx -40.85$$

(5) The moment method for inv. exponential distribution,

$$E(X^k)=\theta^k\Gamma(1-k),\quad k<1$$

$$\therefore E(X^{-1})=\theta^{-1}\Gamma(2)=\theta^{-1}$$

$$\because E(X^{-1})=\frac{1}{n}\sum_{j=1}^{n}x_j^{-1}$$

$$=y=0.0118060483$$

$\therefore\ \hat{\theta}\approx 84.7$ which is the same as the MLE.

(6) The moment method for inv. gamma distribution,

$$E(X^k) = \frac{\theta^k \Gamma(\alpha - k)}{\Gamma(\alpha)}, \quad k < \alpha$$

$$\therefore E(X) = \frac{\theta^1 \Gamma(\alpha - 1)}{\Gamma(\alpha)}$$

At $\alpha = 2$, $E(X) = \theta \dfrac{\Gamma(1)}{\Gamma(2)} = \theta \quad \therefore \theta = E(X) = \dfrac{1}{7}\sum_{j=1}^{7} x_j = 129.86$

which differs from the MLE.