King Saud University
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Final Exam, S2-1443H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

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## Answer the following questions:

**(Note that SND Table is attached in page 3)**

## Q1: [4+4]

(a) The cdf of a random variable $X$ is $F(x) = 1 - \exp\left(-\frac{x}{\theta}\right), \; x > 0$.

Determine the mean excess loss and limited expected value functions.

(b) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and variance of the gamma distribution.

## Q2: [2+6]

(a) Let $X$ have cdf $F_X(x) = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$. Determine the inverse distribution of $X$, with clarifying, the names of distributions.

(b) A sample of 100 losses revealed that 62 were below 1,000 and 38 were above 1,000. An exponential distribution with mean $\theta$ is considered. Using only the given information, determine the maximum likelihood estimate of $\theta$. Now suppose you are also given that the 62 losses that were below 1,000 totaled 28,140, while the total for the 38 above 1,000 remains unknown. Using this additional information, determine the maximum likelihood estimate of $\theta$.

## Q3: [4+4]

(a) The average claim size for a group of insureds is 1,500, with a standard deviation of 7,500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) A group of insureds had 6,000 claims and a total loss of 15,600,000. The prior estimate of the total loss was 16,500,000. Determine the limited fluctuation credibility estimate of the total loss for the group. Use the standard for full credibility determined in (a).

## Q4: [8]

The amount of a claim $X$ has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter $\Theta$ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(a) The probability models for $X$, and risk parameter $\Theta$.

(b) The predictive distribution of the fourth claim and the posterior distribution of $\Theta$.

(c) The Bayesian premium.

(d) The Bühlmann premium.

**Hint**: For Bühlmann premium use, $\mu = \frac{\beta}{\alpha - 1}$, $\upsilon = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$,

where $\beta$ here is the reciprocal of the usual scale parameter of gamma distribution.

## Q5: [8]

Suppose that the number of claims from $m_j$ policies is $N_j$ in year $j$ for a group policyholder with risk parameter $\Theta$ has a Poisson distribution with mean $m_j\Theta$, that is, for $j = 1, \ldots, n$,

$$\Pr(N_j = x \mid \Theta = \theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!}, \quad x = 0, 1, 2, \ldots,$$

where $\Theta$ has a gamma distribution with parameters $\alpha$ and $\beta$. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the $m_{n+1}$ policies.

## Standard Normal Cumulative Probability Table


**Cumulative probabilities for POSITIVE z-values are shown in the following table:**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

# The Model Answer

## Q1: [4+4]

(a)

The mean excess function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)}$$

$$\because S(x) = \exp\left(-\frac{x}{\theta}\right)$$

$$\Rightarrow e_X(x) = \frac{\int_x^\infty \exp\left(-\frac{t}{\theta}\right)dt}{\exp\left(-\frac{x}{\theta}\right)}$$

$$\therefore e_X(x) = \frac{-\theta.\exp\left(-\frac{t}{\theta}\right)\Big|_x^\infty}{\exp\left(-\frac{x}{\theta}\right)} = \theta$$

$$\because E(X \wedge x) = E(X) - e(x)S(x)$$

$$\therefore E(X \wedge x) = \theta - \theta.\exp\left(-\frac{x}{\theta}\right)$$

$$= \theta(1 - e^{-x/\theta})$$

(b)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x)e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^\alpha},$$

where $r(\theta) = -1/\theta$, $\mathrm{q}(\theta) = \theta^\alpha$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean,

$$E(X) = \mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$$
$$= \frac{\alpha\theta^{\alpha-1}}{1/\theta^2.\theta^\alpha} = \alpha\theta$$

and the variance,

$$Var(X) = v(\theta)$$
$$= \frac{\mu'(\theta)}{r'(\theta)}$$
$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

## Q2: [2+6]

(a)

For loglogistic distribution with 2 parameters $\gamma$ and $\theta$, $F_X(x) = \dfrac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$

For $Y = X^{-1}$, we have $F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \frac{(y^{-1}/\theta)^\gamma}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{1}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{(y\theta)^\gamma}{1+(y\theta)^\gamma}$$

Which is also loglogistic distribution with 2 parameters $\gamma$ and $\theta$, $\gamma$ unchanged and $\theta \to 1/\theta$.

(b)

For the first part of the pb, we have

$$L(\theta) = \left[F(1000)\right]^{62}\left[1-F(1000)\right]^{38}$$
$$= \left[1-e^{-1000/\theta}\right]^{62}\left[e^{-1000/\theta}\right]^{38}$$

Let $x = e^{-1000/\theta}$, then

$$L(x) = (1-x)^{62} x^{38}$$

$\Rightarrow$

$$l(x) = 62\ln(1-x) + 38\ln x$$

$\Rightarrow$

$$l^{'}(x) = \frac{-62}{1-x} + \frac{38}{x}$$

Set $l^{'}(x) = 0$, then $\dfrac{-62x + 38(1-x)}{x(1-x)} = 0$

$$\therefore \; x = 0.38$$

$\Rightarrow$

$$0.38 = e^{-1000/\theta}$$

$$\therefore \; \hat{\theta} = -1000 / \ln 0.38 = 1033.50$$

For the second part of the pb (additional information), we have

$$L(\theta) = \left[\prod_{j=1}^{62} f(x_j)\right]\left[S(1000)\right]^{38}$$

$$= \theta^{-62} e^{-28,140/\theta} e^{-38,000/\theta}$$

$$L(\theta) \; = \theta^{-62} e^{-66,140/\theta}$$

$\Rightarrow$

$$l(\theta) = -62\ln(\theta) - 66,140/\theta$$

$$\therefore \; l^{'}(\theta) = \frac{-62}{\theta} + \frac{66,140}{\theta^2}$$

$\Rightarrow$

$$-62\theta^2 + 66,140\theta = 0$$

$$\therefore \; \hat{\theta} = 1,066.77$$

## Q3: [4+4]

(a)

at $p = 0.90$, $\Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1+(\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = 7500^2$, $\theta = 1500$

$\therefore$ The expected # of claims $= 1082.41[1+\left(\tfrac{7500}{1500}\right)^2]$

$$= 28142.66$$

(b)

The credibility factor is $Z = \sqrt{\frac{n}{\lambda_0 \sigma^2/\xi^2}}$

$$= \sqrt{\frac{6000}{28142.66}} = 0.461735$$

The partial credibility through premium is

$$P_c = Z\overline{X} + (1-Z)M$$
$$= 0.461735(15600000) + (1-0.461735)(16500000)$$
$$= 16084438.5$$

## Q4: [8]

(a)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\, \Gamma(4)}, \qquad \theta > 0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta} (1000)^4}{6}, \qquad \theta > 0$$

(b)

The marginal density at the observed values is

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta)\right]\pi(\theta)d\theta$$

$$f(100, 950, 450) = \int_0^\infty \theta e^{-100\theta} \theta e^{-950\theta} \theta e^{-450\theta}\, \frac{1000^4}{6} \theta^3 e^{-1000\theta} d\theta$$

$$f(100,950,450)=\frac{1000^4}{6}\int_0^\infty \theta^6 e^{-2500\theta}d\theta$$

Let $t=2500\theta$

$$f(100,950,450)=\frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6}e^{-t}\frac{dt}{2500}$$

$$f(100,950,450)=\frac{1000^4}{6(2500)^7}\int_0^\infty t^6 e^{-t}dt$$

$$\therefore f(100,950,450)=\frac{1000^4}{6(2500)^7}\Gamma(7)$$

$$\therefore f(100,950,450)=\frac{1000^4}{6}\frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4)=\int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$=\frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$=\frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4\,|100,950,450)=\frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e

$$f(x_4\,|100,950,450)=\frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameter 7 and 2500.

To get posterior density of $\Theta$ given $\mathrm{X}$, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})=\frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta} \ \frac{1000^4}{6}\theta^3 e^{-1000\theta}}{\frac{1000^4}{6}\frac{720}{(2500)^7}}$$

$$= \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(c) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}|\mathrm{X=x}) = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4|100,950,450) = \int_0^\infty \frac{1}{\theta}\cdot\ \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}d\theta$$

$$= \frac{(2500)^7}{720}\int_0^\infty \theta^5 e^{-2500\theta}d\theta$$

$$= \frac{2500}{720}\int_0^\infty u^5 e^{-u}du,\ u = 2500\theta$$

$$= \frac{2500}{720}\Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4|100,950,450) = 416.67$$

(d) To get the Bühlmann premium,

The hypothetical mean is $\mu(\Theta) = \Theta^{-1}$ where $\Theta \sim gamma(4{,}1000)$, $\beta = 1000$ (the reciprocal of the usual scale parameter) and $\alpha = 4$.

$\mu = E[\mu(\Theta)] = E(\Theta^{-1}) = \dfrac{\beta}{\alpha - 1} = \dfrac{1{,}000}{3}$ is the expected value of hypothetical means,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta^{-2}) = \dfrac{\beta^2}{(\alpha-1)(\alpha-2)} = \dfrac{500{,}000}{3}$ is the expected value of process variance and

$a = Var(\Theta^{-1}) = \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \dfrac{500{,}000}{9}$ is the variance of hypothetical means.

We can also find $a$ as $a = Var(\Theta^{-1}) = \frac{500,000}{3} - \left(\frac{1,000}{3}\right)^2 = \frac{500,000}{9}$

$\therefore k = \frac{\upsilon}{a} = 3$, Z$= \frac{n}{n+k} = \frac{3}{6} = \frac{1}{2}$.

So, the Bühlmann Premium is

$$P_c = Z\overline{X} + (1-Z)\mu$$

$$\therefore P_c = \frac{1}{2}(500) + \frac{1}{2}(\frac{1000}{3}) = 416.67, \quad \text{where } \overline{X} = \frac{100+950+450}{3} = 500,$$

which matches the result of Bayesian premium that introduced in (c).

## Q5: [8]

Let $X_j = N_j / m_j$ be the average of claims per individual in year $j$.

$\because N_j|\Theta$ has a Poisson distribution with mean $m_j\Theta$,

$$E(X_j|\Theta) = E\left(\frac{N_j}{m_j}\middle|\Theta\right) = \frac{m_j\Theta}{m_j} = \Theta = \mu(\Theta)$$

and

$$Var(X_j|\Theta) = Var\left(\frac{N_j}{m_j}\middle|\Theta = \theta\right)$$

$$= \frac{1}{m_j^2} Var(N_j|\Theta) = \frac{m_j\Theta}{m_j^2}$$

$$= \frac{\Theta}{m_j} = \frac{\upsilon(\Theta)}{m_j}$$

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical means, where $\Theta \sim gamma(\alpha, \beta)$,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$\therefore k = \frac{\upsilon}{a} = \frac{1}{\beta}$, Z$= \frac{m}{m+k} = \frac{m\beta}{m\beta+1}$.

So, the Bühlmann-Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$

$$= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X} = \mathrm{m}^{-1}\sum_{j=1}^{n} m_j X_j$$

For year $n+1$, the estimate is $m_{n+1}P_c$.