King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2-1442H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 20

Answer the following questions:

(Note that SND Table is attached in page 3)

Q1: [6+3]

(a) If the random variable X has probability density function $f(x) = (1+2x^2)e^{-2x},\ x \ge 0.$

(i) Determine the survival function.

(ii) Determine the hazard rate function.

(iii) Determine the mean excess loss function.

(b) The cdf of a random variable is $F(x) = 1 - x^{-2},\ x \ge 1.$ Determine the mean, median, and mode of this random variable.

Q2: [4+2]

(a) Let $X|\Lambda$ have a Poisson distribution with parameter Λ. Let Λ have a Gamma distribution with parameters α and θ. Determine the expectation of Λ and then the unconditional expectation of X.

(b) Consider a frailty model with frailty random variable Λ, such that $a(x) = \frac{1}{x+1},\quad x > 0.$

Find the conditional survival function of X.

Q3: [5+5]

(a) The average claim size for a group of insureds is 1500, with a standard deviation of 7500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) A group of insureds had 6000 claims and a total loss of 15,600,000. The prior estimate of the total loss was 16,500,000. Determine the limited fluctuation credibility estimate of the total loss for the group. Use the standard for full credibility determined in (a).

Q4: [5+5]

(a) There are two types of drivers. Good drivers make up 70% of the population and in one year have zero claims with probability 0.6, one claim with probability 0.3, and two claims with probability 0.1. Bad drivers make up the other 30% of the population and have zero, one, or two claims with probabilities 0.4, 0.3, and 0.3, respectively.

(i) Describe this process by using the concept of the risk parameter Θ.

(ii) For a particular policyholder, suppose that we have observed $x_1 = 0$ and $x_2 = 1$ for past claims. Determine the posterior distribution of $\Theta | X_1 = 0,\ X_2 = 1$ and the predictive distribution of $X_3 | X_1 = 0,\ X_2 = 1.$

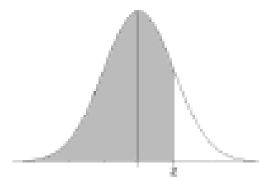
(b)

Risk 1 produces claims of amounts 100, 1,000, and 20,000 with probabilities 0.5, 0.3, and 0.2, respectively. For risk 2, the probabilities are 0.7, 0.2 and 0.1. Risk 1 is twice as likely as risk 2 of being observed. A claim of 100 is observed, but the observed risk is unknown. Determine the Buhlmann credibility estimate of the expected value of the second claim amount from the same risk.

Q5: [5]

Polices have a deductible of 7. Six losses are observed, with value 10, 13, 14, 15, 17, 19. Ground-up losses have an exponential distribution with parameter θ. Determine the maximum likelihood estimate of θ.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [6+3]

(a)

(i)

The survival function is

$$S(x) = \int_x^\infty (1+2t^2)e^{-2t}dt$$

$$= -\tfrac{1}{2}e^{-2t} + 2I, \text{ where } I = \int_x^\infty t^2 e^{-2t}dt$$

$$I = \int t^2 e^{-2t}dt$$

$$= -\tfrac{1}{4}e^{-2t} - \tfrac{t}{2}e^{-2t} - \tfrac{t^2}{2}e^{-2t}$$

$$\therefore\ S(x) = -(1+t+t^2)e^{-2t}\Big|_x^\infty$$

$$= (1+x+x^2)e^{-2x},\ x \ge 0$$

(ii)

$\because$ The hazard rate function is

$$h(x) = -\frac{d}{dx}[\ln S(x)]$$

and $\because S(x) = (1+x+x^2)e^{-2x}$

$$\ln S(x) = -2x + \ln(1+x+x^2)$$

$$\therefore\ h(x) = 2 - \frac{1+2x}{1+x+x^2}$$

or simply,

$$h(x) = \frac{f(x)}{S(x)} = \frac{1+2x^2}{1+x+x^2}$$

(iii)

The mean excess loss function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)} \qquad (1)$$

From (i) $S(x) = (1+x+x^2)e^{-2x}$ (2)

We can deduce that,

$$
\begin{aligned}
\int_x^\infty S(t)dt &= \int_x^\infty (1+t+t^2)e^{-2t}dt \\
&= -(1+t+\tfrac{1}{2}t^2)e^{-2t}\Big|_x^\infty \\
&= (1+x+\tfrac{1}{2}x^2)e^{-2x} \qquad (3)
\end{aligned}
$$

Where $I = \int t^2 e^{-2t}dt = -\frac{1}{4}e^{-2t} - \frac{t}{2}e^{-2t} - \frac{t^2}{2}e^{-2t}$,

$\int te^{-2t}dt = -\frac{1}{4}e^{-2t} - \frac{t}{2}e^{-2t}$ and $\int e^{-2t}dt = -\frac{1}{2}e^{-2t}$

$\therefore$ By substituting (2) and (3) in (1), we get

$$
e_X(x) = \frac{1+x+\frac{1}{2}x^2}{1+x+x^2}
$$

(b)

The pdf is $f(x) = 2x^{-3},\ x \geq 1.$

The mean is

$$
\begin{aligned}
E(X) &= \int_1^\infty 2x^{-2}dx \\
&= 2
\end{aligned}
$$

To get the median, solve the equation $F(x) = 1 - x^{-2} = 0.5$

$\Rightarrow$ The median $\simeq 1.4142$

The mode is the value at which the pdf is highest, so to get the mode,

$\because$ $f(x) = 2x^{-3},\ x \geq 1$ is a decreasing function and its highest value at $x = 1$

$\therefore$ The mode =1

Q2: [4+2]

(a)

$\because \ \Lambda \sim$ Gamma (α, θ)

$\therefore \ f_\Lambda(\lambda) = \dfrac{(\lambda/\theta)^\alpha e^{-\lambda/\theta}}{\lambda \Gamma(\alpha)}$

$\therefore \ E(\Lambda) = \alpha\theta$

$\because \ X|\Lambda \ \sim$ Poisson (Λ)

$\therefore \ E(X|\Lambda) = \Lambda$

$\because \ E(X) = E\left[E(X|\Lambda)\right] = E(\Lambda)$

$\therefore \ E(X) = \alpha\theta = E(\Lambda)$

Note: Also, you can obtain this result by using the formula

$$f_X(x) = \int_0^\infty f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda) d\lambda \text{ where } f_{X|\Lambda}(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

(b)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\,dt$$
$$= \int_0^x \frac{dt}{1+t} = \ln(1+x)$$

Thus,

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$
$$= e^{-\lambda \ln(1+x)}$$
$$= \frac{1}{(1+x)^\lambda}$$

Q3: [5+5]

(a)

at $p = 0.90$, $\Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1+(\tfrac{\sigma}{\theta})^2]$$

where σ^2=7500^2, θ=1500

$\therefore$ The expected # of claims $=1082.41[1+\left(\tfrac{7500}{1500}\right)^2]$

$$=28142.66$$

(b)

The credibility factor is $Z=\sqrt{\frac{n}{\lambda_0\sigma^2/\xi^2}}$

$$=\sqrt{\frac{6000}{28142.66}}=0.461735$$

The partial credibility through premium is

$$P_c = Z\overline{X}+(1-Z)M$$

$$=0.461735(15600000)+(1-0.461735)(16500000)$$

$$=16084438.5$$

Q4: [5+5]

(a)

(i)

x	$\Pr(X=x \mid \Theta=G)$	$\Pr(X=x \mid \Theta=B)$	θ	$\Pr(\Theta=\theta)$
0	0.6	0.4	G	0.7
1	0.3	0.3	B	0.3
2	0.1	0.3		

(ii)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G|0,1) = \frac{f(0|G)f(1|G)\pi(G)}{f_X(0,1)}$$

where $f_X(0,1) = \sum_\theta f_{X_1|\Theta}(0|\theta) f_{X_2|\Theta}(1|\theta)\pi(\theta)$

$$\begin{aligned} f_X(0,1) &= 0.6(0.3)(0.7) + 0.4(0.3)(0.3) \\ &= 0.162 \end{aligned}$$

$$\pi(G|0,1) = \frac{0.6(0.3)(0.7)}{0.162} = 0.7778$$

$$\pi(B|0,1) = \frac{0.4(0.3)(0.3)}{0.162} = 0.2222$$

For the predictive distribution, the predictive probabilities are given by

$$\begin{aligned} f_{X_3|X}(0|0,1) &= \sum_\theta f(0|\theta)\pi(\theta|0,1) \\ &= f(0|G)\pi(G|0,1) + f(0|B)\pi(B|0,1) \\ &= 0.6(0.7778) + 0.4(0.2222) \\ &= 0.55556, \end{aligned}$$

$$\begin{aligned} f_{X_3|X}(1|0,1) &= \sum_\theta f(1|\theta)\pi(\theta|0,1) \\ &= f(1|G)\pi(G|0,1) + f(1|B)\pi(B|0,1) \\ &= 0.3(0.7778) + 0.3(0.2222) \\ &= 0.3, \end{aligned}$$

and $f_{X_3|X}(2|0,1) = \sum_\theta f(2|\theta)\pi(\theta|0,1)$

$$\begin{aligned} &= f(2|G)\pi(G|0,1) + f(2|B)\pi(B|0,1) \\ &= 0.1(0.7778) + 0.3(0.2222) \\ &= 0.14444. \end{aligned}$$

(b)

The required calculations are given in the following table.

Risk	100	1,000	20,000	$\mu(\Theta)$	$v(\Theta)$	$\Pr(\Theta = \theta)$
1	0.5	0.3	0.2	4,350	61,382,500	2/3
2	0.7	0.2	0.1	2,270	35,054,100	1/3

To determine the Buhlmann credibility estimate of the expected value of the second claim amount from the same risk, we should find the following quantities.

$\mu = E[\mu(\Theta)]$

$=(2/3)(4350)+(1/3)(2270)=3{,}656.667,$

$v = E[v(\Theta)]$

$=(2/3)(61{,}382{,}500)+(1/3)(35{,}054{,}100)=52{,}606{,}366.67,$

$a = \text{var}[\mu(\Theta)]$

$= (2/3)(4350)^2 + (1/3)(2270)^2 - 3{,}656.667^2 = 961{,}419.7845,$

$k = \dfrac{v}{a}$

$= 54.71737,$

$Z = \dfrac{n}{n+k}$

$= \dfrac{1}{1+54.71737}$

$\therefore\ Z = \dfrac{1}{55.71737} = 0.0179477.$

The Buhlmann estimate is

$$E(X_2|100) = P_c = Z\overline{X} + (1-Z)\mu$$
$$= 0.0179477(100) + (1-0.0179477)(3{,}656.667)$$
$$= 3{,}592.83.$$

where $X_1 = 100$

Q5: [5]

The likelihood function is

$$L(\theta)=\prod_{j=1}^{6}\frac{f(x_j|\theta)}{1-F(7|\theta)}$$

$$=\frac{f(10|\theta)f(13|\theta)f(14|\theta)f(15|\theta)f(17|\theta)f(19|\theta)}{[1-F(7|\theta)]^6}$$

$$=\frac{\frac{1}{\theta^6}e^{-1/\theta(10+13+14+15+17+19)}}{\left[e^{-7/\theta}\right]^6}$$

$$\therefore\quad L(\theta)=\tfrac{1}{\theta^6}e^{-46/\theta}$$

The loglikelihood function is

$$l(\theta)=-\frac{46}{\theta}-6\ln\theta$$

To get $\hat{\theta}$, set $l'(\theta)=0$

$$\Rightarrow l'(\theta)=\frac{46}{\theta^2}-\frac{6}{\theta}=0$$

$$\therefore\quad \hat{\theta}=\frac{46}{6}$$

$$\simeq 7.6667$$

Another Solution

$$L(\theta)=\prod_{j=1}^{6}\frac{f(x_j|\theta)}{1-F(7|\theta)}=\frac{\prod_{j=1}^{6}\frac{1}{\theta}e^{-x_j/\theta}}{(e^{-7/\theta})^6}$$

The loglikelihood function is

$$l(\theta)=-6\ln\theta-\frac{1}{\theta}\sum_{j=1}^{6}x_j+6\left(\frac{7}{\theta}\right)$$

To get $\hat{\theta}$, set $l'(\theta)=0$

$$\Rightarrow l'(\theta)=-\frac{6}{\theta}+\frac{88}{\theta^2}-\frac{42}{\theta^2}=0 \text{ where } \sum_{j=1}^{6}x_j=88$$

$$l'(\theta)=-\frac{6}{\theta}+\frac{46}{\theta^2}=0$$

$$\therefore\quad \hat{\theta}=\frac{46}{6}$$

$$\simeq 7.6667$$