King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1-1447H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: 7[1+1+1+2+2]

Determine the survival, density, hazard rate, mean excess loss and limited expected value functions for the age at death model that defined as:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.01x, & 0 \le x < 100 \\ 1, & x \ge 100 \end{cases}$$

Q2: 7[3+4]

(a) Let X have cdf $F_X(x) = \dfrac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}$. Determine the inverse distribution of X, with clarifying, the names of distributions.

(b) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and variance of the gamma distribution.

Hint: For gamma distribution $f(x|\alpha,\theta) = \dfrac{(x/\theta)^\alpha e^{-x/\theta}}{x\,\Gamma(\alpha)}$.

Q3: 8[2+2+2+2]

The amount of a claim X has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(i) The probability density functions for X, and risk parameter Θ.

(ii) The predictive distribution of the fourth claim and the posterior distribution of Θ.

(iii) The Bayesian premium.

(iv) The Bühlmann premium.

Hint: For Bühlmann premium estimate in (iv) use, $\mu = \dfrac{\beta}{\alpha-1}$, $\upsilon = \dfrac{\beta^2}{(\alpha-1)(\alpha-2)}$, where β is the reciprocal of the usual scale parameter of gamma distribution.

Q4: 9[4+5]

(a) The average claim size for a group of insureds is 1,500, with a standard deviation of 7,500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that the number of claims from m_j policies is N_j in year j for a group policyholder with risk parameter Θ has a Poisson distribution with mean $m_j\Theta$, that is, for $j = 1, \ldots, n$,

$$\Pr(N_j = x | \Theta = \theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!}, \quad x = 0, 1, 2, \ldots,$$

where Θ has a gamma distribution with parameters α and β. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the m_{n+1} policies.

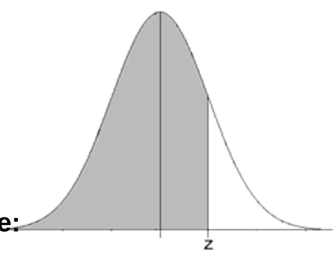
Q5: 9[3+6]

(a) Demonstrate that the Weibull distribution is a scale distribution.

Hint: For Weibull distribution, $F(x) = 1 - e^{-(x/\theta)^\tau}$.

(b) A random sample of size 5 is taken from a Weibull distribution with $\tau = 2$. Two of the sample observations are known to exceed 50 and the three remaining observations are 20, 30, and 45. Determine the maximum likelihood estimate of θ. Also, find the value of the log-likelihood function.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: 7[1+1+1+2+2]

(i) $S(x) = 1 - 0.01x,\ 0 \le x < 100$

(ii) $f(x) = F^{'}(x) = -S^{'}(x) = 0.01$

(iii)

$$\because h(x) = \frac{f(x)}{S(x)}$$

$$\therefore h(x) = \frac{0.01}{1-0.01x}$$

(v)

$$\begin{aligned}\because e_X(d) &= \frac{\int_d^{\infty} S(x)dx}{S(d)} \\ &= \frac{\int_d^{100}(1-0.01x)dx}{1-0.01d} \\ &= \frac{-1}{0.01(1-0.01d)}\left[\frac{(1-0.01x)^2}{2}\right]_d^{100}\end{aligned}$$

$$\therefore e_X(d) = \frac{1-0.01d}{0.02} = \frac{100-d}{2}$$

(vi)

$$\begin{aligned}\because E(X \wedge u) &= \int_{-\infty}^{u} xf(x)dx + u[1-F(u)] \\ &= \int_0^u x(0.01)dx + u(1-0.01u) \\ &= 0.005u^2 + u - 0.01u^2\end{aligned}$$

$$\therefore E(X \wedge u) = u(1-0.005u)$$

or

$$E(X \wedge u) = \int_0^u S(x)dx$$

$$= \int_0^u (1 - 0.01x)dx$$

$$= \left[x - \frac{0.01x^2}{2} \right]_0^u$$

$$\therefore E(X \wedge u) = u - 0.005u^2$$

or

By using the following formula

$$E(X \wedge u) = E(X) - e(u)S(u)$$

$$\because E(X) = \int_0^u x(0.01)dx = \int_0^{100} x(0.01)dx$$

$$= 0.01 \left[\frac{x^2}{2} \right]_0^{100}$$

$$\therefore \mathrm{E(x)} = 50$$

Also, $e(u) = \frac{100-u}{2}$ and $S(u) = 1 - 0.01u$

$$\because E(X \wedge u) = E(X) - e(u)S(u)$$

$$= 50 - (\tfrac{100-u}{2})(1 - 0.01u)$$

$$= 50 - \tfrac{0.01}{2}(100 - u)^2$$

$$\therefore E(X \wedge u) = u - 0.005u^2$$

Q2: 7[3+4]

(a)

For loglogistic distribution with 2 parameters γ and θ, $F_X(x) = \dfrac{(x/\theta)^\gamma}{1 + (x/\theta)^\gamma}$

For $Y = X^{-1}$, we have $F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \frac{(y^{-1}/\theta)^\gamma}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{1}{1+(y^{-1}/\theta)^\gamma}$$
$$= \frac{(y\theta)^\gamma}{1+(y\theta)^\gamma}$$

Which is also loglogistic distribution with 2 parameters γ and θ, γ unchanged and $\theta \to 1/\theta$.

(b)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \dfrac{p(x) e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^\alpha},$$

where $r(\theta) = -1/\theta$, $\mathrm{q}(\theta) = \theta^\alpha$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean,

$$E(X) = \mu(\theta) = \frac{q^{'}(\theta)}{r^{'}(\theta) q(\theta)}$$
$$= \frac{\alpha\theta^{\alpha-1}}{1/\theta^2.\ \theta^\alpha} = \alpha\theta$$

and the variance,

$$Var(X) = v(\theta)$$
$$= \frac{\mu^{'}(\theta)}{r^{'}(\theta)}$$
$$= \frac{\alpha}{1/\theta^2} = \alpha\theta^2$$

Q3: 8[2+2+2+2]

(a)

(i)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\, \Gamma(4)}, \qquad \theta>0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta} (1000)^4}{6}, \qquad \theta>0$$

(ii)

The marginal density at the observed values is

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta) \right] \pi(\theta) d\theta$$

$$f(100, 950, 450) = \int_0^\infty \theta e^{-100\theta} \theta e^{-950\theta} \theta e^{-450\theta} \frac{1000^4}{6} \theta^3 e^{-1000\theta} d\theta$$

$$f(100, 950, 450) = \frac{1000^4}{6} \int_0^\infty \theta^6 e^{-2500\theta} d\theta$$

Let $t = 2500\theta$

$$f(100, 950, 450) = \frac{1000^4}{6} \int_0^\infty \frac{t^6}{(2500)^6} e^{-t} \frac{dt}{2500}$$

$$f(100, 950, 450) = \frac{1000^4}{6(2500)^7} \int_0^\infty t^6 e^{-t} dt$$

$$\therefore f(100, 950, 450) = \frac{1000^4}{6(2500)^7} \Gamma(7)$$

$$\therefore f(100, 950, 450) = \frac{1000^4}{6} \frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4)=\int_0^{\infty}\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$=\frac{1000^4}{6}\int_0^{\infty}\theta^7 e^{-(2500+x_4)\theta}d\theta$$

$$=\frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$=\frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4\,|100,950,450)=\frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e.

$$f(x_4\,|100,950,450)=\frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameter 7 and 2500.

To get posterior density of Θ given X, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})=\frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore\ \pi(\theta|100,950,450)=\frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}}{\frac{1000^4}{6}\frac{720}{(2500)^7}}$$

$$=\frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(iii) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore\ \mu_4(\theta)=\theta^{-1}=\frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}\,|\mathrm{X{=}x})=\int\mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta\,|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4 | 100, 950, 450) = \int_0^\infty \frac{1}{\theta} \cdot \frac{\theta^6 e^{-2500\theta} (2500)^7}{720} d\theta$$

$$= \frac{(2500)^7}{720} \int_0^\infty \theta^5 e^{-2500\theta} d\theta$$

$$= \frac{2500}{720} \int_0^\infty u^5 e^{-u} du, \quad u = 2500\theta$$

$$= \frac{2500}{720} \Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4 | 100, 950, 450) = 416.67$$

Another solution

$$\because f(x_4 | 100, 950, 450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density function with parameters $(\alpha, \theta) = (7, 2500)$

$$\therefore E(x_4 | 100, 950, 450) = \frac{\theta}{\alpha - 1} = \frac{2500}{6} = 416, 67.$$

(iv) To get the Bühlmann premium,

The hypothetical mean is $\mu(\Theta) = \Theta^{-1}$ where $\Theta \sim gamma(4, 1000)$, $\alpha = 4$ and $\beta = 1000$ (the reciprocal of the usual scale parameter).

$\mu = E[\mu(\Theta)] = E(\Theta^{-1}) = \dfrac{\beta}{\alpha - 1} = \dfrac{1,000}{3}$ is the expected value of hypothetical mean,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta^{-2}) = \dfrac{\beta^2}{(\alpha - 1)(\alpha - 2)} = \dfrac{500,000}{3}$ is the expected value of process variance and

$a = Var(\Theta^{-1}) = \dfrac{\beta^2}{(\alpha - 1)^2 (\alpha - 2)} = \dfrac{500,000}{9}$ is the variance of hypothetical mean.

We can also find a as $a = Var(\Theta^{-1}) = \dfrac{500,000}{3} - \left(\dfrac{1,000}{3}\right)^2 = \dfrac{500,000}{9}$

$$\therefore k = \frac{\upsilon}{a} = 3, \ \mathrm{Z} = \frac{n}{n + k} = \frac{3}{6} = \frac{1}{2}.$$

So, the Bühlmann Premium is

$$P_c = Z\overline{X} + (1-Z)\mu$$

$$\therefore P_c = \frac{1}{2}(500) + \frac{1}{2}(\frac{1000}{3}) = 416.67, \quad \text{where } \overline{X} = \frac{100+950+450}{3} = 500,$$

which matches the result of Bayesian premium that introduced in (iii).

Q4: 9[4+5]

(a)

$$\text{at } p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$$
$$\Rightarrow y_p = 1.645 \text{ (by using SND table)}$$
$$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1+(\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = 7500^2$, $\theta = 1500$ \

$$\therefore \text{ The expected \# of claims } = 1082.41[1+\left(\tfrac{7500}{1500}\right)^2]$$
$$= 28{,}142.66$$

(b)

Let $X_j = N_j / m_j$ be the average of claims per individual in year j.

$\because N_j | \Theta$ has a Poisson distribution with mean $m_j\Theta$,

$$E(X_j|\Theta) = E\left(\frac{N_j}{m_j}\middle|\Theta\right) = \frac{m_j\Theta}{m_j} = \Theta = \mu(\Theta)$$

and

$$Var(X_j|\Theta) = Var\left(\frac{N_j}{m_j}\middle|\Theta = \theta\right)$$
$$= \frac{1}{m_j^2}Var(N_j|\Theta) = \frac{m_j\Theta}{m_j^2}$$
$$= \frac{\Theta}{m_j} = \frac{\upsilon(\Theta)}{m_j}$$

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical means, where $\Theta \sim gamma(\alpha, \beta)$,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$\therefore k = \frac{\upsilon}{a} = \frac{1}{\beta}$, Z$= \frac{m}{m+k} = \frac{m\beta}{m\beta+1}$.

So, the Bühlmann-Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$
$$= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X} = \text{m}^{-1}\sum_{j=1}^{n} m_j X_j$$

For year $n+1$, the estimate is $m_{n+1}P_c$.

Q5: 9[3+6]

(a) For the Weibull distribution, $X \sim$ Weibull (θ, τ)

and the distribution function is $F_X(x) = 1 - e^{-(x/\theta)^\tau}$

let $Y = cX$, c > 0, then

$$F_Y(y) = pr(Y \le y)$$
$$= pr(X \le \tfrac{y}{c})$$

$$\therefore \; F_Y(y) = 1 - e^{-(y/c\theta)^\tau}$$

which is a Weibull distribution with parameters τ and $c\theta$

$\therefore$ θ is a scale parameter.

$\therefore$ The Weibull distribution is a scale distribution

(b) For Weibull distribution

$$F(x) = 1 - e^{-(x/\theta)^2}, f(x) = \frac{2x}{\theta^2}e^{-(x/\theta)^2}$$

The likelihood function is

$L(\theta) = f(20)f(30)f(45)[1 - F(50)]^2$

$\therefore L(\theta) = \frac{40}{\theta^2} e^{-(20/\theta)^2} \frac{60}{\theta^2} e^{-(30/\theta)^2} \frac{90}{\theta^2} e^{-(45/\theta)^2} \left[e^{-(50/\theta)^2}\right]^2$

$= 216{,}000\theta^{-6} e^{-8325/\theta^2}$

$\therefore l(\theta) = ln\, 216{,}000 - 6\, ln\, \theta - 8325\theta^{-2}$

which is known as log-likelihood function, to get $\hat{\theta}$ Set $l'(\theta) = 0$

$\Rightarrow -6\theta^{-1} + 2(8325)\theta^{-3} = 0$

$\therefore 6\theta^2 = 16650$

$\therefore \hat{\theta} = \sqrt{\frac{16650}{6}} \simeq 52.68$

$\therefore \ l(\hat{\theta}) \simeq -14.5$