King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1-1446H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [2+2+2+1]

Determine the mean excess loss, limited expected value and probability density functions for the following model.

$$F(x)=\begin{cases}0, & x<0\\ 1-\left(\frac{2{,}000}{x+2{,}000}\right)^3, & x\geq 0\end{cases}$$

and show that this model is a member of the transformed beta family.

Note that: the pdf of generalized beta is defined as $f(x)=\dfrac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}}$.

Q2: [4+3]

(a) One hundred observed claims in 2021 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 10% per year. Based on the empirical distribution from 2021, determine a range for the probability that a claim exceeds 500 in 2024.

(b) Show that the gamma distribution is a member of the linear exponential family, then derive the mean and variance of the gamma distribution.

Hint: For gamma distribution $f(x|\alpha,\theta)=\dfrac{(x/\theta)^{\alpha}e^{-x/\theta}}{x\,\Gamma(\alpha)}$.

Q3: [3+3]

Let a random sample of 6 insurance payments for a random variable X is given as 3, 6, 7, 8, 10, 12.
Find the **mean** of X and the **value** of the log-likelihood function in each of the following two cases.
(a) If X is assumed to have an exponential distribution
(b) If X is assumed to have a gamma distribution with $\alpha=6$.

Q4: [5+5]

(a) The average claim size for a group of insureds is 1,500, with a standard deviation of 7,500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that the number of claims from m_j policies is N_j in year j for a group policyholder with risk parameter Θ has a Poisson distribution with mean $m_j\Theta$, that is, for $j=1,\ldots,n,$

$$\Pr(N_j = x|\Theta=\theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!}, \ x=0,1,2,\ldots,$$

where Θ has a gamma distribution with parameters α and β. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the m_{n+1} policies.

Q5: [5+5]

Risk 1 produces claims of amounts 100, 1,000, and 20,000 with probabilities 0.5, 0.3, and 0.2, respectively. For risk 2, the probabilities are 0.7, 0.2 and 0.1. Risk 1 is twice as likely as risk 2 of being observed. A claim of 100 is observed, but the observed risk is unknown.

(a) Determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk.

(b) Determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [2+2+2+1]

(i)

The mean excess loss function is

$$e_X(d) = \frac{\int_d^\infty S(x)dx}{S(d)}, \quad S(x) = \left(\tfrac{2000}{x+2000}\right)^3$$

$$\therefore e_X(d) = \frac{\int_d^\infty \left(\tfrac{2000}{x+2000}\right)^3 dx}{\left(\tfrac{2000}{d+2000}\right)^3}$$

$$= \tfrac{2000+d}{2}$$

(ii)

To get the limited expected value function $E(X \wedge u)$

$$E(X \wedge u) = -\int_{-\infty}^{0} F(x)dx + \int_{0}^{u} S(x)dx$$

$$\Rightarrow \quad E(X \wedge u) = 0 + \int_0^u \left(\tfrac{2000}{x+2000}\right)^3 dx$$

$$= (2000)^3 \left[\frac{(x+2000)^{-2}}{-2}\right]_0^u$$

$$\therefore \quad E(X \wedge u) \; = 1000\left[1 - \frac{4{,}000{,}000}{(u+2000)^2}\right]$$

(iii)

To get the pdf of the given model

$$f(x) = F^{'}(x)$$

$$= \tfrac{3(2000)^3}{(x+2000)^4}, \quad x > 0$$

Which is the pdf of the Pareto distribution.

(iv)

For $X \sim$ Transformed beta $(\alpha, \theta, \gamma, \tau)$ generalized beta

$$f(x) = \frac{\Gamma(\alpha+\tau)\gamma(x/\theta)^{\gamma\tau}}{\Gamma(\alpha)\Gamma(\tau)x[1+(x/\theta)^{\gamma}]^{\alpha+\tau}} \quad (1)$$

$at\ \gamma = \tau = 1$

$$(1) \Rightarrow f(x) = \frac{\Gamma(\alpha+1)(x/\theta)}{\Gamma(\alpha)\Gamma(1)x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha!(x/\theta)}{(\alpha-1)!x[1+(x/\theta)]^{\alpha+1}}$$

$$= \frac{\alpha}{\theta} / \left(\frac{x+\theta}{\theta}\right)^{\alpha+1}$$

$\therefore f(x) = \dfrac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}$ which is a Pareto Prob. density function (2)

$\therefore$ The given model where $\alpha = 3,\ \theta = 2000$ is a member of the transformed beta family.

Q2: [4+3]

(a)

The amount in 2021	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 10% per year

In 2022 $\rightarrow$ 1.1 *X*, in 2023 $\rightarrow$ 1.21 *X* and in 2024 $\rightarrow$ 1.331 *X*

where *X* is the random variable of the claim in 1995 and *Y*=1.331 *X* is the random variable of the claim in 2024. $\Pr(Y > 500) = \Pr(X > 500/1.331) = \Pr(X > 376)$

From given data, $\Pr(X > 350)$ =55/100=0.55 and $\Pr(X > 400)$ =50/100=0.50

$\therefore\ 0.50 < \Pr(Y > 500) < 0.55$

(b)

For $X \sim gamma(\alpha, \theta)$

$$\Rightarrow f(x;\theta) = \frac{\theta^{-\alpha} x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}$$

Clearly, $f(x;\theta) = \frac{p(x) e^{r(\theta)x}}{q(\theta)}$

$$= \frac{[x^{\alpha-1}/\Gamma(\alpha)].\ e^{-\frac{1}{\theta}x}}{\theta^{\alpha}},$$

where $r(\theta) = -1/\theta,\ \mathrm{q}(\theta) = \theta^{\alpha}$ and $p(x) = x^{\alpha-1}/\Gamma(\alpha)$

$\therefore$ The gamma distribution is a member of the linear exponential family.

$\therefore$ The mean,

$$E(X) = \mu(\theta) = \frac{q^{'}(\theta)}{r^{'}(\theta) q(\theta)}$$

$$= \frac{\alpha \theta^{\alpha-1}}{1/\theta^{2}.\ \theta^{\alpha}} = \alpha\theta$$

and the variance,

$$Var(X) = v(\theta)$$

$$= \frac{\mu^{'}(\theta)}{r^{'}(\theta)}$$

$$= \frac{\alpha}{1/\theta^{2}} = \alpha\theta^{2}$$

Q3: [3+3]

(a)

The likelihood function is

$$L(\theta) = \prod_{j=1}^{6} f(x_j | \theta)$$

$$= f(3|\theta) f(6|\theta) f(7|\theta) f(8|\theta) f(10|\theta) f(12|\theta)$$

$$= \tfrac{1}{\theta^{6}} e^{-1/\theta(3+6+7+8+10+12)}$$

$$\therefore\ L(\theta) = \tfrac{1}{\theta^{6}} e^{-46/\theta}$$

The loglikelihood function is

$$l(\theta) = -\frac{46}{\theta} - 6\ln\theta$$

To get $\hat{\theta}$, set $l'(\theta) = 0$

$$\Rightarrow l'(\theta) = \frac{46}{\theta^2} - \frac{6}{\theta} = 0$$

$$\therefore \ \hat{\theta} = \frac{46}{6}$$

$$\simeq 7.6667$$

$$\therefore \ E(X) = \theta = 7.6667$$

The loglikelihood function is

$$l(\theta) = -\frac{46}{\theta} - 6\ln\theta$$

$$\therefore l(\hat{\theta}) = -\frac{46}{7.6667} - 6\ln 7.6667$$

$$= -18.2213$$

(b)

For a gamma distribution with $\alpha = 6$, $f(x_j \mid \theta) = \dfrac{x_j^5 \theta^{-6} e^{-x_j/\theta}}{\Gamma(6)}$, $j = 1, 2, \ldots, 6$

$$l(\theta) = \sum_{j=1}^{n} \ln(f(x_j \mid \theta)) = \sum_{j=1}^{6} \ln(\frac{x_j^5 \theta^{-6} e^{-x_j/\theta}}{120})$$

$$= 5\sum_{j=1}^{6} \ln x_j - 36\ln\theta - \theta^{-1}\sum_{j=1}^{6} x_j - 6\ln(120)$$

Set $l'(\theta) = 0$

$$\Rightarrow -36\theta^{-1} + \theta^{-2}\sum_{j=1}^{6} x_j = 0$$

$$-36\theta^{-1} + 6\theta^{-2}\bar{x} = 0 \qquad (\times \tfrac{\theta^2}{36})$$

$$\therefore \ \hat{\theta} = \tfrac{\bar{x}}{6}$$

$$\therefore \ \hat{\theta} = 1.2778$$

$\therefore\ E(X) = \alpha\theta = 6(1.2778) \simeq 7.67$

$$\therefore l(\theta) = 5(11.7032) - 36\ln(1.2778) - \left(\frac{46}{1.2778}\right) - 28.7250$$

$$= -15.0334$$

Q4: [5+5]

(a)

at $p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = 7500^2,\ \theta = 1500$ \

$\therefore$ The expected # of claims $= 1082.41[1 + \left(\tfrac{7500}{1500}\right)^2]$

$= 28{,}142.66$

(b)

Let $X_j = N_j / m_j$ be the average of claims per individual in year j.

$\because N_j | \Theta$ has a Poisson distribution with mean $m_j\Theta$,

$$E(X_j|\Theta) = E\left(\frac{N_j}{m_j}\middle|\Theta\right) = \frac{m_j\Theta}{m_j} = \Theta = \mu(\Theta)$$

and

$$Var(X_j|\Theta) = Var\left(\frac{N_j}{m_j}\middle|\Theta = \theta\right)$$

$$= \frac{1}{m_j^2} Var(N_j|\Theta) = \frac{m_j\Theta}{m_j^2}$$

$$= \frac{\Theta}{m_j} = \frac{\upsilon(\Theta)}{m_j}$$

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical means, where $\Theta \sim gamma(\alpha, \beta)$,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$$\therefore k = \frac{\upsilon}{a} = \frac{1}{\beta}, \text{ Z=}\frac{m}{m+k} = \frac{m\beta}{m\beta+1}.$$

So, the Bühlmann-Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$
$$= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X}=\text{m}^{-1}\sum_{j=1}^{n} m_j X_j$$

For year $n+1$, the estimate is $m_{n+1}P_c$.

Q5: [5+5]

The required calculations are given in the following table.

Risk	100	1,000	20,000	$\mu(\Theta)$	$v(\Theta)$	$\Pr(\Theta=\theta)$
1	0.5	0.3	0.2	4,350	61,382,500	2/3
2	0.7	0.2	0.1	2,270	35,054,100	1/3

(a) To determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk, we do the following.

Clearly, $\pi(\theta=1) = \frac{2}{3}, \ \pi(\theta=2) = \frac{1}{3}$

The marginal probability is $f_X(x) = \sum_{\theta} f(x|\theta)\pi(\theta)$

$$f(100) = f(100|1)\pi(1) + f(100|2)\pi(2)$$
$$= 0.5\left(\frac{2}{3}\right) + 0.7\left(\frac{1}{3}\right) = \frac{17}{30}$$

The posterior probabilities are given by

$$\pi(1|100) = \frac{f(100|1)\pi(1)}{f(100)} = \frac{10}{17} \text{ and } \pi(2|100) = 1 - \frac{10}{17} = \frac{7}{17}$$

The hypothetical means are

$$\mu(1) = 4350, \ \mu(2) = 2270$$

The expected next value through Bayesian premium is

$$E(X_2|100) = \pi(1|100)\mu(1) + \pi(2|100)\mu(2)$$
$$= 3,493.53$$

where $X_1 = 100$.

(b) To determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk, we should find the following quantities.

$$\mu = E[\mu(\Theta)]$$
$$=(2/3)(4350)+(1/3)(2270)=3,656.667,$$

$$v = E[v(\Theta)]$$
$$=(2/3)(61,382,500)+(1/3)(35,054,100)=52,606,366.67,$$

$$a = \text{var}[\mu(\Theta)]$$
$$= (2/3)(4350)^2 + (1/3)(2270)^2 - 3,656.667^2 = 961,419.7845,$$

$$k = \frac{v}{a}$$
$$= 54.71737,$$

$$Z = \frac{n}{n+k}$$
$$= \frac{1}{1+54.71737}$$
$$\therefore \ Z = \frac{1}{55.71737} = 0.0179477.$$

The Bühlmann estimate is

$$
\begin{aligned}
E(X_2 \mid 100) = P_c &= Z\overline{X} + (1-Z)\mu \\
&= 0.0179477(100) + (1-0.0179477)(3,656.667) \\
&= 3,592.83.
\end{aligned}
$$

where $X_1 = 100$.