King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1-1444H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [3+4]

(a) For the model of automobile bodily injury claim that is defined by an insurance company as

$$F(x)=\begin{cases}0, & x<0,\\ 1-\left(\dfrac{2000}{x+2000}\right)^3, & x\geq 0.\end{cases}$$

Determine the survival, density, and hazard rate functions.

(b) The cdf of a random variable X is $F(x)=1-\exp\left(-\dfrac{x}{\theta}\right),\ x>0.$

Find $e_X(x)$ and $E(X\wedge x)$.

Q2: [7]

Let X have a Burr distribution with parameters $\alpha=1,\ \gamma=2$ and $\theta=\sqrt{1000}$ and let Y have a Pareto distribution with parameters $\alpha=1$ and $\theta=1000.$ Let Z be a mixture of X and Y with equal weight on each component. Determine the median of Z. Let $W=1.1\ Z$. Demonstrate that W is also a mixture of a Burr and a Pareto distribution, and determine the parameters of W.

Q3: [5+5]

(a) An insurance company has decided to establish its full-credibility requirements for an individual state rate filing. The full-credibility standard is to be set so that the observed total amount of claims underlying the rate filing would be within 5% of the true value with probability 0.90. The claim frequency follows a Poisson distribution and the severity distribution has pdf

$$f(x)=\frac{100-x}{5{,}000},\quad 0\leq x\leq 100$$

Determine the expected number of claims necessary to obtain full credibility using the normal approximation.

(b) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table

Year	1	2	3
Claims	475	550	400

Determine the full credibility and partial credibility through premium by assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

Q4: [5+5]

Risk 1 produces claims of amounts 100, 1,000, and 20,000 with probabilities 0.5, 0.3, and 0.2, respectively. For risk 2, the probabilities are 0.7, 0.2 and 0.1. Risk 1 is twice as likely as risk 2 of being observed. A claim of 100 is observed, but the observed risk is unknown.

(a) Determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk.

(b) Determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk.

Q5: [6]

Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimate of the parameter θ for an exponential distribution, and for a gamma distribution with $\alpha = 2$. Also, find the value of the log-likelihood function in each case.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [3+4]

(a) The survival function is

$$S(x) = -F(x)$$

$$\therefore\ S(x) = \left(\frac{2000}{x+2000}\right)^3,\ x \geq 0$$

The density function is

$$f(x) = F^{'}(x) = -S^{'}(x)$$

$$\therefore\ f(x)\ = \frac{3(2000)^3}{(x+2000)^4},\ x > 0$$

The hazard rate function

$$h(x) = \frac{f(x)}{S(x)}$$

$$\therefore\ h(x) = \frac{3}{(x+2000)},\ x > 0$$

(b)

The mean excess function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)}$$

$$\because S(x) = \exp\left(-\frac{x}{\theta}\right)$$

$$\Rightarrow e_X(x) = \frac{\int_x^\infty \exp\left(-\frac{t}{\theta}\right)dt}{\exp\left(-\frac{x}{\theta}\right)}$$

$$\therefore e_X(x) = \frac{-\theta.\exp\left(-\frac{t}{\theta}\right)\Big|_x^\infty}{\exp\left(-\frac{x}{\theta}\right)} = \theta$$

$$\because E(X \wedge x) = E(X) - e(x)S(x)$$

$$\therefore E(X \wedge x) = \theta - \theta.\exp\left(-\frac{x}{\theta}\right)$$

$$= \theta(1 - e^{-x/\theta})$$

Q2: [7]

$$\because X \sim Burr\text{ - }\alpha = 1,\ \gamma = 2,\ \theta = \sqrt{1000}$$

$$\therefore F(x) = 1 - u^{\alpha},\ u = \frac{1}{1 + (x/\theta)^{\gamma}}$$

$$= 1 - \frac{1}{1 + (x/\sqrt{1000})^2}$$

and $\because Y \sim Pareto\text{ - }\alpha = 1$ and $\theta = 1000$

$$F(y) = 1 - \left(\frac{\theta}{y + \theta}\right)^{\alpha}$$

$$= 1 - \left(\frac{1}{1 + y/\theta}\right)^{\alpha}$$

$$\therefore F(y) = 1 - \frac{1}{1 + y/1000}$$

For mixture distribution

$$F_Z(z) = 0.5\left(1 - \frac{1}{1 + (z/\sqrt{1000})^2}\right) + 0.5\left(1 - \frac{1}{1 + z/1000}\right) \qquad (1)$$

$$= 1 - 0.5\frac{1000}{1000 + z^2} - 0.5\frac{1000}{1000 + z}$$

$$\therefore F_Z(z) = 1 - \frac{0.5[2(1000)^2 + 1000z + 1000z^2]}{(1000 + z^2)(1000 + z)} \qquad (2)$$

The median m is the solution of the equation $F(m) = 0.5$ (3)

Apply (3) in (2), we get

$$\frac{0.5[2(1000)^2 + 1000m + 1000m^2]}{(1000 + m^2)(1000 + m)} = 0.5$$

$$\Rightarrow m^3 = (1000)^2 = 10^6$$

$\therefore m = 100$ which is the median.

For $W = 1.1Z$

$$F_W(w) = \Pr(W \le w) = \Pr(1.1Z \le w) = \Pr(Z \le \frac{w}{1.1})$$

$$F_W(w) = F_Z(w/1.1) = 0.5\left(1 - \frac{1}{1 + (w/1.1\sqrt{1000})^2}\right) + 0.5\left(1 - \frac{1}{1 + w/1100}\right)$$

Which is a 50/50 mixture of a Burr distribution with parameters $\alpha = 1,\ \gamma=2$ and $\theta = 1.1\sqrt{1000}$ and a Pareto distribution with parameters $\alpha = 1$ and $\theta = 1100$.

Q3: [5+5]

(a)

$$\text{at } p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$$
$$\Rightarrow y_p = 1.645 \text{ (by using SND table)}$$
$$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$$

$$E(X) = \int_0^{100} x\left(\frac{100-x}{5000}\right) dx$$
$$= \int_0^{100} \frac{100x - x^2}{5,000}\, dx$$
$$= \frac{1}{5,000}\left[100(x^2/2) - x^3/3\right]_0^{100}$$
$$\therefore E(X) = \frac{100^3}{5,000}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{100}{3}$$
$$E(X^2) = \int_0^{100} x^2\left(\frac{100-x}{5,000}\right) dx$$
$$= \int_0^{100} \frac{100x^2 - x^3}{5,000}\, dx = \frac{5,000}{3}$$
$$\therefore Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{5,000}{3} - \frac{10,000}{9} = \frac{5,000}{9}$$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = \dfrac{5,000}{9}$, $\theta^2 = \dfrac{10,000}{9}$

$$\therefore \text{ The expected \# of claims } = 1082.41[1 + 0.5]$$
$$= 1623.615$$

(b)

at $p = 0.95,\ \Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

The mean is $\xi = E(X_j) = \dfrac{475+550+400}{3} = 475,$

variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)^2}{n-1} = \dfrac{0^2 + 75^2 + 75^2}{2} = 5625$

For full credibility $n \geq \lambda_0 \left(\dfrac{\sigma}{\xi}\right)^2$

$\therefore n \geq 1536.64\left(\dfrac{5625}{475^2}\right)$

$\therefore n \geq 38.3095845$

The credibility factor is $Z = \sqrt{\dfrac{n}{\lambda_0 \sigma^2/\xi^2}}$

$= \sqrt{\dfrac{3}{38.3095845}} = 0.279838$

The partial credibility through premium is

$P_c = Z\overline{X} + (1-Z)M$

$= 0.279838(475) + (1-0.279838)(600)$

$\therefore P_c = 565.02025$

Q4: [5+5]

The required calculations are given in the following table.

Risk	100	1,000	20,000	$\mu(\Theta)$	$v(\Theta)$	$\Pr(\Theta = \theta)$
1	0.5	0.3	0.2	4,350	61,382,500	2/3
2	0.7	0.2	0.1	2,270	35,054,100	1/3

(a) To determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk, we do the following.

Clearly, $\pi(\theta=1)=\frac{2}{3},\ \pi(\theta=2)=\frac{1}{3}$

The marginal probability is $f_X(x)=\sum_{\theta} f(x|\theta)\pi(\theta)$

$$f(100)=f(100|1)\pi(1)+f(100|2)\pi(2)$$
$$=0.5\left(\frac{2}{3}\right)+0.7\left(\frac{1}{3}\right)=\frac{17}{30}$$

The posterior probabilities are given by

$$\pi(1|100)=\frac{f(100|1)\pi(1)}{f(100)}=\frac{10}{17} \text{ and } \pi(2|100)=1-\frac{10}{17}=\frac{7}{17}$$

The hypothetical means are

$$\mu(1)=4350,\ \mu(2)=2270$$

The expected next value through Bayesian premium is

$$E(X_2|100)=\pi(1|100)\mu(1)+\pi(2|100)\mu(2)$$
$$=3493.53$$

where $X_1=100$.

(b) To determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk, we should find the following quantities.

$$\mu=E[\mu(\Theta)]$$
$$=(2/3)(4350)+(1/3)(2270)=3{,}656.667,$$

$$v=E[v(\Theta)]$$
$$=(2/3)(61{,}382{,}500)+(1/3)(35{,}054{,}100)=52{,}606{,}366.67,$$

$$a=\text{var}[\mu(\Theta)]$$
$$=(2/3)(4350)^2+(1/3)(2270)^2 - 3{,}656.667^2=961{,}419.7845,$$

$$k=\frac{v}{a}$$
$$=54.71737,$$

$$Z = \frac{n}{n+k}$$

$$= \frac{1}{1+54.71737}$$

$$\therefore \text{ Z } = \frac{1}{55.71737} = 0.0179477.$$

The Bühlmann estimate is

$$E(X_2 | 100) = P_c = Z\overline{X} + (1-Z)\mu$$
$$= 0.0179477(100) + (1-0.0179477)(3,656.667)$$
$$= 3,592.83.$$

where $X_1 = 100.$

Q5: [6]

For exponential distribution, the likelihood function is

$$L(\theta) = f(27)f(82)f(115)f(126)f(155)f(161)f(243)$$
$$= \theta^{-1}e^{-27/\theta}\theta^{-1}e^{-82/\theta}\theta^{-1}e^{-115/\theta}\theta^{-1}e^{-126/\theta}\theta^{-1}e^{-155/\theta}\theta^{-1}e^{-161/\theta}\theta^{-1}e^{-243/\theta}$$
$$= \theta^{-7}e^{-909/\theta}$$

$$\therefore l(\theta) = -7\ln\theta - 909\theta^{-1}$$

which is known as log-likelihood function, to get the likelihood estimate of the parameter θ

Set $l'(\theta) = 0$

$$\Rightarrow -7\theta^{-1} + 909\theta^{-2} = 0$$

$\therefore \hat{\theta} = 129.85714$ which is the MLE of the mean of an exponential model.

$$\therefore l(\hat{\theta}) = -41.065$$

For a gamma distribution with $\alpha = 2,$ $f(x_j | \theta) = x_j\theta^{-2}e^{-x_j/\theta}, \ j = 1, 2, ..., 7$

$$\therefore l(\theta) = \sum_{j=1}^{n} \ln(f(x_j | \theta)) = \sum_{j=1}^{7} \ln(x_j\theta^{-2}e^{-x_j/\theta})$$
$$= \sum_{j=1}^{7} \ln x_j - 14\ln\theta - \theta^{-1}\sum_{j=1}^{7} x_j$$

Set $l^{'}(\theta) = 0$

$$\Rightarrow -14\theta^{-1} + \theta^{-2}\sum_{j=1}^{7} x_j = 0$$

$$-14\theta^{-1} + 7\theta^{-2}\bar{x} = 0 \qquad (\times \frac{\theta^2}{14})$$

$$\therefore \hat{\theta} = \frac{\bar{x}}{2}$$

$$\therefore \hat{\theta} = 64.9286$$

$$\Rightarrow l(\hat{\theta}) = \sum_{j=1}^{7} \ln x_j - 14\ln\hat{\theta} - 14$$

$$\therefore l(\hat{\theta}) = -39.5244$$
