King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1-1443H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions:

(Note that SND Table is attached in page 3)

Q1: [6+3]

(a) Actuaries at an Insurance Services Office, considered a mixture of two Pareto distributions as follows

$$F(x) = 1 - a\left(\frac{\theta_1}{\theta_1 + x}\right)^{\alpha} - (1-a)\left(\frac{\theta_2}{\theta_2 + x}\right)^{\alpha+2}$$

Determine the mean and variance of this mixture distribution.

(b) The cdf of a random variable X is $F(x) = 1 - \exp\left(-\frac{x}{\theta}\right),\ x > 0.$

Find $e_X(x)$ and $E(X \wedge x)$.

Q2: [4+2]

Consider the exponential-inverse Gaussian frailty model with

$$a(x) = \frac{\theta}{2\sqrt{1+\theta x}},\ \theta > 0$$

(a) Determine the conditional survival function $S_{X|\Lambda}(x|\lambda)$.

(b) If Λ has a gamma distribution with parameters $\theta = 1$ and α replaced by 2α, determine the marginal or unconditional survival function of X.

Q3: [5+5]

(a) An insurance company has decided to establish its full-credibility requirements for an individual state rate filing. The full-credibility standard is to be set so that the observed total amount of claims underlying the rate filing would be within 5% of the true value with probability 0.90. The claim frequency follows a Poisson distribution and the severity distribution has pdf

$$f(x) = \frac{100 - x}{5{,}000},\quad 0 \le x \le 100$$

Determine the expected number of claims necessary to obtain full credibility using the normal approximation.

(b) For a particular policyholder, the manual premium is 600 per year. The past claims experience is given in the following table

Year	1	2	3
Claims	475	550	400

Determine the full credibility and partial credibility through premium by assuming the normal approximation. Use $r = 0.05$ and $p = 0.95$.

Q4: [5+5]

(a) There are two types of drivers. Good drivers make up 75% of the population and in one year have zero claims with probability 0.8, one claim with probability 0.1, and two claims with probability 0.1. Bad drivers make up the other 25% of the population and have zero, one, or two claims with probabilities 0.6, 0.2, and 0.2, respectively.

(i) Describe this process by using the concept of the risk parameter Θ.

(ii) For a particular policyholder, suppose that we have observed $x_1 = 0$ and $x_2 = 1$ for past claims. Determine the posterior distribution of $\Theta | X_1 = 0,\ X_2 = 1$ and the predictive distribution of $X_3 | X_1 = 0,\ X_2 = 1.$

(b) Claim sizes have an exponential distribution with mean θ. For 80% of risks, $\theta = 8$, and for 20% of risks, $\theta = 2$. A randomly selected policy had a claim of size 5 in year 1. Determine both the Bayesian and Bühlmann estimates of the expected claim size in year 2.

Q5: [5]

A ground up loss X has a deductible of 7 applied. A random sample of 6 insurance payments (after deductible is applied) is given as 3, 6, 7, 8, 10, 12. If X is assumed to have an exponential distribution, apply maximum likelihood estimation to estimate the mean of X and the value of the log-likelihood function.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [6+3]

(a)

For the mixture of 2 Pareto distributions

$$F(x)=1-a\left(\frac{\theta_1}{\theta_1+x}\right)^{\alpha}-(1-a)\left(\frac{\theta_2}{\theta_2+x}\right)^{\alpha+2}$$

The m^{th} moment of a k-point mixture distribution is given by

$$\because E(Y^m)=\int y^m[a_1 f_{X_1}(y)+\ldots+a_k f_{X_k}(y)]dy$$

$$\therefore\ E(Y^m)\ =a_1E(Y_1^m)+\ldots+a_kE(Y_k^m)$$

For $m=1$ and two point mixture distribution

$$\Rightarrow\ E(Y)\ =aE(Y_1)+(1-a)E(Y_2)$$

For Pareto - (α,θ)

$$E(X^k)=\frac{\theta^k k!}{(\alpha-1)\ldots(\alpha-k)}$$, where k is a positive integer

$$\Rightarrow\ E(X)=\frac{\theta}{(\alpha-1)},\ \alpha>1 \text{ and } E(X^2)=\frac{2\theta^2}{(\alpha-1)(\alpha-2)},\ \alpha>2$$

$\therefore$ The mean is given by

$$E(Y)\ =a\frac{\theta_1}{\alpha-1}+(1-a)\frac{\theta_2}{\alpha+2-1}$$

$$E(Y)\ =a\frac{\theta_1}{\alpha-1}+(1-a)\frac{\theta_2}{\alpha+1},\ \alpha>1$$

Similarly, for the second moment

$$E(Y^2)\ =aE(Y_1^2)+(1-a)E(Y_2^2)$$

$$E(Y^2)=a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)}+(1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)},\ \alpha>2$$

$$\text{Variance} = E(Y^2) - [E(Y)]^2$$

$$= a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)} + (1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)} - a^2\frac{\theta_1^2}{(\alpha-1)^2} - (1-a)^2\frac{\theta_2^2}{(\alpha+1)^2} - 2a(1-a)\frac{\theta_1\theta_2}{(\alpha^2-1)}$$

$$\text{Variance} = a\frac{2\theta_1^2}{(\alpha-1)(\alpha-2)} - a^2\frac{\theta_1^2}{(\alpha-1)^2} + (1-a)\frac{2\theta_2^2}{\alpha(\alpha+1)} - (1-a)^2\frac{\theta_2^2}{(\alpha+1)^2} - 2a(1-a)\frac{\theta_1\theta_2}{(\alpha^2-1)}$$

(b)

The mean excess function is

$$e_X(x) = \frac{\int_x^\infty S(t)dt}{S(x)}$$

$$\because S(x) = \exp\left(-\frac{x}{\theta}\right)$$

$$\Rightarrow e_X(x) = \frac{\int_x^\infty \exp\left(-\frac{t}{\theta}\right)dt}{\exp\left(-\frac{x}{\theta}\right)}$$

$$\therefore e_X(x) = \frac{-\theta.\exp\left(-\frac{t}{\theta}\right)\Big|_x^\infty}{\exp\left(-\frac{x}{\theta}\right)} = \theta$$

$$\because E(X \wedge x) = E(X) - e(x)S(x)$$

$$\therefore E(X \wedge x) = \theta - \theta.\exp\left(-\frac{x}{\theta}\right)$$

$$= \theta(1 - e^{-x/\theta})$$

Q2: [4+2]

(a)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\, dt$$
$$= \int_0^x \frac{\theta}{2\sqrt{1+\theta t}} dt$$
$$= \frac{1}{2}\int_0^x (1+\theta t)^{-\frac{1}{2}} \theta dt$$
$$\therefore A(x) = \sqrt{1+\theta x} - 1$$

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$
$$= e^{-\lambda(\sqrt{1+\theta x}-1)}$$

(b)

$\because \Lambda \sim gamma\ (2\alpha, 1)$

$\therefore$ The moment generating function of the frailty random variable Λ is

$$M_\Lambda(z) = E(e^{z\Lambda})$$
$$= \left(\frac{1}{1-z}\right)^{2\alpha} = (1-z)^{-2\alpha}$$

The marginal survival function is

$$S_X(x) = E(e^{-\Lambda A(x)})$$
$$= M_\Lambda[-A(x)]$$
$$\therefore S_X(x) = (1+\sqrt{1+\theta x}-1)^{-2\alpha}$$
$$= (1+\theta x)^{-\alpha}$$

Which is a Pareto distribution.

Q3: [5+5]

(a)

at $p = 0.90,\ \Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.645/0.05)^2 = 1082.41$

$$E(X) = \int_0^{100} x\left(\frac{100-x}{5000}\right) dx$$
$$= \int_0^{100} \frac{100x - x^2}{5,000}\ dx$$
$$= \frac{1}{5,000}\left[100(x^2/2) - x^3/3\right]_0^{100}$$
$$\therefore E(X) = \frac{100^3}{5,000}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{100}{3}$$
$$E(X^2) = \int_0^{100} x^2\left(\frac{100-x}{5,000}\right) dx$$
$$= \int_0^{100} \frac{100x^2 - x^3}{5,000}\ dx = \frac{5,000}{3}$$
$$\therefore Var(X) = E(X^2) - [E(X)]^2$$
$$= \frac{5,000}{3} - \frac{10,000}{9} = \frac{5,000}{9}$$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\tfrac{\sigma}{\theta})^2]$$

where $\sigma^2 = \dfrac{5,000}{9}$, $\theta^2 = \dfrac{10,000}{9}$

$\therefore$ The expected # of claims $= 1082.41[1 + 0.5]$

$= 1623.615$

(b)

at $p = 0.95$, $\Phi(y_p) = (1+p)/2 = 0.975$

$\Rightarrow y_p = 1.96$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p / r)^2 = (1.96/0.05)^2 = 1536.64$

The mean is $\xi = E(X_j) = \dfrac{475 + 550 + 400}{3} = 475,$

variance is $\sigma^2 = \dfrac{\sum_j (x_j - \xi)^2}{n-1} = \dfrac{0^2 + 75^2 + 75^2}{2} = 5625$

For full credibility $n \ge \lambda_0 \left(\dfrac{\sigma}{\xi}\right)^2$

$\therefore n \ge 1536.64\left(\dfrac{5625}{475^2}\right)$

$\therefore n \ge 38.3095845$

The credibility factor is $Z=\sqrt{\frac{n}{\lambda_0\sigma^2/\xi^2}}$

$$=\sqrt{\frac{3}{38.3095845}}=0.279838$$

The partial credibility through premium is

$$P_c=Z\overline{X}+(1-Z)M$$
$$=0.279838(475)+(1-0.279838)(600)$$
$$\therefore P_c=565.02025$$

Q4: [5+5]

(a)

(i)

x	$\Pr(X=x\|\Theta=G)$	$\Pr(X=x\|\Theta=B)$	θ	$\Pr(\Theta=\theta)$
0	0.8	0.6	G	0.75
1	0.1	0.2	B	0.25
2	0.1	0.2		

(ii)

For the posterior distribution, the posterior probabilities are given by

$$\pi(G|0,1)=\frac{f(0|G)f(1|G)\pi(G)}{f_X(0,1)}$$

where $f_X(0,1)=\sum_{\theta} f_{X_1|\Theta}(0|\theta)f_{X_2|\Theta}(1|\theta)\pi(\theta)$

$$f_X(0,1)=0.8(0.1)(0.75)+0.6(0.2)(0.25)$$
$$=0.09$$
$$\pi(G|0,1)=\frac{0.8(0.1)(0.75)}{0.09}\simeq 0.67$$
$$\pi(B|0,1)=\frac{0.6(0.2)(0.25)}{0.09}\simeq 0.33$$

For the predictive distribution, the predictive probabilities are given by

$$f_{X_3|X}(0|0,1) = \sum_{\theta} f(0|\theta)\pi(\theta|0,1)$$
$$= f(0|G)\pi(G|0,1) + f(0|B)\pi(B|0,1)$$
$$= 0.8(0.67) + 0.6(0.33)$$
$$= 0.734,$$

$$f_{X_3|X}(1|0,1) = \sum_{\theta} f(1|\theta)\pi(\theta|0,1)$$
$$= f(1|G)\pi(G|0,1) + f(1|B)\pi(B|0,1)$$
$$= 0.1(0.67) + 0.2(0.33)$$
$$= 0.133,$$

and $f_{X_3|X}(2|0,1) = \sum_{\theta} f(2|\theta)\pi(\theta|0,1)$
$$= f(2|G)\pi(G|0,1) + f(2|B)\pi(B|0,1)$$
$$= 0.1(0.67) + 0.2(0.33)$$
$$= 0.133.$$

(b)

The Bayesian estimate of the expected claim size in year 2.

We have $\pi(\Theta = 8) = 0.80$ and $\pi(\Theta = 2) = 0.20$, and # of claims (claim size) is 5 in year 1.

$$E(X_2|X_1 = 5) = E(\Theta|X_1 = 5)$$
$$= \mu(\Theta{=}8)\pi(\Theta = 8|X_1 = 5) + \mu(\Theta{=}2)\pi(\Theta = 2|X_1 = 5)$$

$$\pi(\Theta = 8|X_1 = 5) = \frac{\Pr(X_1 = 5|\Theta = 8)\pi(\Theta = 8)}{\Pr(X_1 = 5|\Theta = 8)\pi(\Theta = 8) + \Pr(X_1 = 5|\Theta = 2)\pi(\Theta = 2)}$$
$$= \frac{(1/8)e^{-5/8}(0.8)}{(1/8)e^{-5/8}(0.8) + (1/2)e^{-5/2}(0.2)} = 0.867035$$

Similarly, $\pi(\Theta = 2|X_1 = 5) = \dfrac{\Pr(X_1 = 5|\Theta = 2)\pi(\Theta = 2)}{\Pr(X_1 = 5|\Theta = 8)\pi(\Theta = 8) + \Pr(X_1 = 5|\Theta = 2)\pi(\Theta = 2)}$
$$= \frac{(1/2)e^{-5/2}(0.2)}{(1/8)e^{-5/8}(0.8) + (1/2)e^{-5/2}(0.2)} = 0.132965$$

$$\therefore E(X_2|X_1 = 5) = 8 \times 0.867035 + 2 \times 0.132965$$
$$= 7.2022$$

The Bühlmann estimate of the expected claim size in year 2.

To determine the Bühlmann credibility estimate, we should find the following quantities.

$$\mu = E[\mu(\Theta)]$$
$$= 8(0.80) + (2)(0.20) = 6.8,$$

$$a = \text{var}[\mu(\Theta)]$$
$$= 8^2(0.8) + 2^2(0.2) - 6.8^2 = 5.76,$$

$$v = E[v(\Theta)]$$
$$= \sum_{\theta} v(\theta)\pi(\theta)$$
$$= 8^2 \times 0.8 + 2^2 \times 0.2 = 52,$$

Note that for $X \sim \exp(\theta)$ the mean $= E(X) = \theta$ and $\text{var}(X) = \theta^2$

$$k = \frac{v}{a}$$
$$= \frac{52}{5.76} = 9.02778,$$

$$Z = \frac{n}{n+k}$$
$$= \frac{1}{1+9.02778}$$
$$\therefore \ Z = 0.099723.$$

The Bühlmann estimates is

$$E(X_2 | 100) = P_c = Z\overline{X} + (1-Z)\mu$$
$$= 0.099723 \times 5 + (1-0.099723) \times 6.8$$
$$= 6.6205.$$

Q5: [5]

First Method (shifted approach)

The loss amounts after the deductible is applied are: 3, 6, 7, 8, 10, 12

The likelihood function is

$$L(\theta) = \prod_{j=1}^{6} f(x_j|\theta)$$

$$= f(3|\theta)f(6|\theta)f(7|\theta)f(8|\theta)f(10|\theta)f(12|\theta)$$

$$= \tfrac{1}{\theta^6} e^{-1/\theta(3+6+7+8+10+12)}$$

$$\therefore \quad L(\theta) = \tfrac{1}{\theta^6} e^{-46/\theta}$$

The loglikelihood function is

$$l(\theta) = -\frac{46}{\theta} - 6\ln\theta$$

To get $\hat{\theta}$, set $l'(\theta) = 0$

$$\Rightarrow l'(\theta) = \frac{46}{\theta^2} - \frac{6}{\theta} = 0$$

$$\therefore \quad \hat{\theta} = \frac{46}{6}$$

$$\simeq 7.6667$$

Second Method (un-shifted approach)

The loss amounts before the deductible is applied are: 10, 13, 14, 15, 17, 19

The likelihood function is

$$L(\theta) = \prod_{j=1}^{6} \frac{f(x_j|\theta)}{1 - F(7|\theta)}$$

$$= \frac{f(10|\theta)f(13|\theta)f(14|\theta)f(15|\theta)f(17|\theta)f(19|\theta)}{[1 - F(7|\theta)]^6}$$

$$= \frac{\tfrac{1}{\theta^6} e^{-1/\theta(10+13+14+15+17+19)}}{\left[e^{-7/\theta}\right]^6}$$

$$\therefore \quad L(\theta) = \tfrac{1}{\theta^6} e^{-46/\theta}$$

The loglikelihood function is

$$l(\theta) = -\frac{46}{\theta} - 6\ln\theta$$

To get $\hat{\theta}$, set $l^{'}(\theta) = 0$

$$\Rightarrow l^{'}(\theta) = \frac{46}{\theta^2} - \frac{6}{\theta} = 0$$

$$\therefore \quad \hat{\theta} = \frac{46}{6}$$

$$\simeq 7.6667$$

$$\therefore l(\hat{\theta}) = -\frac{46}{7.6667} - 6\ln 7.6667$$

$$= -18.2213$$
