King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2-1447H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: 6[2+2+2]

An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses: 100(0.4), 500(0.2), 1000(0.2), 2500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for the excess loss, and left censored and shifted variables, where the deductible is set at 750. Also, calculate the probability mass function and the expected value of the limited loss variable (right censored variable) with a limit of 750.

Q2: 8[4+4]

(a) Losses in 2023 follow the density function $f(x) = 2x^{-3}, x \geq 1$, where x is the loss in millions of dollars. Inflation of 10% impacts all claims uniformly from 2023 to 2024. Determine the cdf of losses for 2024 and use it to determine the probability that a 2024 loss exceeds 2,200,000.

(b) Let Λ have a gamma distribution and let $X|\Lambda$ have a Weibull distribution with conditional survival function $S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$. Determine the unconditional or marginal distribution of X.

Q3: 6[4+2]

(a) Seven losses are observed as 27, 82, 115, 126, 155, 161 and 243. Determine the maximum likelihood estimates of the parameter θ for the inverse exponential and inverse gamma with $\alpha = 2$ distributions. Also, find the value of the log-likelihood function in each case.

Hint: For inverse gamma distribution $f(x;\theta) = \dfrac{(\theta/x)^{\alpha} e^{-\theta/x}}{x\,\Gamma(\alpha)}$.

(b) Five hundred losses are observed. Five of the losses are 1100, 3200, 3300, 3500, and 3900. All that is known about the other 495 losses is that they exceed 4000. Determine the maximum likelihood estimate of the mean of an exponential model and the value of the log-likelihood function.

Q4: 10[2+2+3+3]

The amount of a claim X has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(i) The probability density functions for X and risk parameter Θ.

(ii) The predictive distribution of the fourth claim and the posterior distribution of Θ.

(iii) The Bayesian premium.

(iv) The Bühlmann premium.

Hint: For Bühlmann premium estimate in (iv) use, $\mu = \frac{\beta}{\alpha-1}$, $\upsilon = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$, where β is the reciprocal of the usual scale parameter of gamma distribution.

Q5: 10[4+6]

(a) The average claim size for a group of insureds is 2500, with a standard deviation of 7500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that, given $\Theta = \theta$, the random variable $X_1, X_2, ..., X_n$ are independent and identical distributed with Poisson probability function, $f_{X_j|\Theta}(x_j|\theta) = \dfrac{\theta^{x_j} e^{-\theta}}{x_j!}$, $x_j = 0,1,2,...$

(i) Let $S = X_1 + X_2 + ... + X_n$. Show that the probability function for S is given by

$$f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!} \pi(\theta) d\theta, \quad s = 0,1,2,...$$

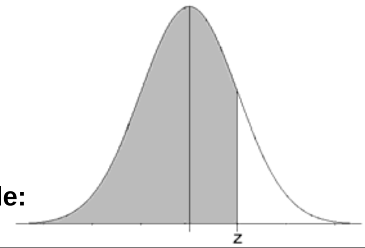
where Θ has pdf $\pi(\theta)$.

(ii) Show that the Bayesian Premium is

$$\mathrm{E}(X_{n+1}|\mathrm{X} = \mathrm{x}) = \frac{1+n\bar{x}}{n} \frac{f_S\left(1+n\bar{x}\right)}{f_S\left(n\bar{x}\right)}$$

(iii) Evaluate the distribution of S when $\Theta \sim Gamma(\alpha, \beta)$, and give its name.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: 6[2+2+2]

The probability of exceeding the deductible is

$$Pr(X > 750) = 1 - F(750), d = 750$$
$$= 0.2 + 0.1 + 0.1 = 0.4$$

For the excess loss variable $Y^p = X - d$, we have

$$\begin{array}{llll} x_j - d: & 250 & 1750 & 9250 \\ \dfrac{p(x_j)}{1 - F(d)}: & 0.2/0.4 & 0.1/0.4 & 0.1/0.4 \\ & 0.5 & 0.25 & 0.25 \end{array}$$

The expected value for the excess loss variable is defined as

$$e_X(d) = E(Y^p) = E(X - d|X > d)$$

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2{,}875$$

For the left censored and shifted variable $Y^L = (X - d)_+$

$$\begin{array}{lllll} (x_j - d)_+: & 0 & 250 & 1750 & 9250 \\ p(x_j): & 0.6 & 0.2 & 0.1 & 0.1 \end{array}$$

The expected value for the left censored and shifted variable is

$$E(Y^L) = E[(X - d)_+], \text{where } Y^L = \begin{cases} 0, & X \le d \\ X - d, & X > d \end{cases}$$

$$\therefore E(Y^L) = E[(X - d)_+]$$
$$= 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$$
$$= 1{,}150$$

For the limited loss variable (right censored variable)

$$\begin{array}{llll} x_j \wedge d: & 100 & 500 & 750 \\ p(x_j): & 0.4 & 0.2 & 0.4 \end{array}$$

The expected value for the limited loss variable (right censored variable) is

$$E(Y) = E(X \wedge d), \text{where } Y = \begin{cases} X, X < d \\ d, X \ge d \end{cases}$$

$$\therefore E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$$
$$= 440$$

Note that: $E(X) = E[(X - d)_+] + E(X \wedge d)$
$$= 1{,}150 + 440$$

$= 1{,}590$

Q2: 8[4+4]

(a)

$$F(x) = \int_1^x 2t^{-3}\, dt$$
$$= 1 - x^{-2}$$

$\because Y = 1.1\,X$

$$F_Y(y) = 1 - \left(\frac{y}{1.1}\right)^{-2}$$

$\therefore \Pr(Y > 2.2) = 1 - F_Y(2.2) = 0.25$

(b)

let $\Lambda \sim$ Gamma (θ, α), $X|\Lambda \sim$ Weibull(λ, γ)

$\therefore\ S_{X|\Lambda}(x|\lambda) = e^{-\lambda x^{\gamma}}$

$\therefore\ A(x) = x^{\gamma}$

$\because\ S_X(x) = E[e^{-\Lambda A(x)}]$
$= M_{\Lambda}[-A(x)]$

$\therefore\ S_X(x) = M_{\Lambda}[-x^{\gamma}]$

and $\because M_{\Lambda}(z) = (1 - \theta z)^{-\alpha}$

$\therefore\ S_X(x) = (1 + \theta x^{\gamma})^{-\alpha}$

which is a Burr distribution with parameters

$\alpha \to \alpha$, $\theta \to \theta^{\frac{-1}{\gamma}}$ and $\gamma \to \gamma$.

Q3: 6[4+2]

(a)

(1) For inv. exponential distribution, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n} \frac{\theta e^{-\theta/x_j}}{x_j^2}$$

$$l(\theta) = \sum_{j=1}^{n} \ln\theta - \sum_{j=1}^{n} \theta / x_j - 2\sum_{j=1}^{n} \ln x_j$$

$$l(\theta) = n\ln\theta - \theta\sum_{j=1}^{n} x_j^{-1} - 2\sum_{j=1}^{n}\ln x_j$$

$\therefore\ l(\theta) = n\ln\theta - ny\theta - 2\sum_{j=1}^{n}\ln x_j$, where $y = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

To get max. estimate of θ (i.e. $\hat{\theta}$), set $l'(\theta) = 0$

$$l'(\theta) = n\theta^{-1} - ny = 0$$

$\Rightarrow\ \theta^{-1} = y$ i.e. $\hat{\theta} = 1/y$

(2) The value of the log-likelihood function for inv. exponential distribution is given by

$$l(\hat{\theta}) = 7\ln\hat{\theta} - 7 - 2\sum_{j=1}^{7}\ln x_j$$

$$= 7\ln(84.702347) - 7 - 2(32.90166107)$$

$$\approx -41.73$$

(3) For inv. gamma distribution with $\alpha = 2$, the likelihood function is

$$L(\theta) = \prod_{j=1}^{n}\theta^2 x_j^{-3} e^{-\theta/x_j}$$

$l(\theta) = 2\sum_{j=1}^{n}\ln\theta - 3\sum_{j=1}^{n}\ln x_j - ny\theta$, where $y = \frac{1}{n}\sum_{j=1}^{n} x_j^{-1}$

$$l(\theta) = 2n\ln\theta - ny\theta - 3\sum_{j=1}^{n}\ln x_j$$

To get $\hat{\theta}$, let $l'(\theta) = 0$

$$l'(\theta) = 2n\theta^{-1} - ny = 0$$

$\Rightarrow\ \theta^{-1} = y/2$ i.e. $\hat{\theta} = 2/y$

$\therefore\ \hat{\theta} \approx 169.4$

(4) The value of the log-likelihood function for inv. gamma distribution is given by

$$l(\hat{\theta}) = 14\ln\hat{\theta} - 14 - 3\sum_{j=1}^{7}\ln x_j$$

$$= 14\ln(169.404694) - 14 - 3(32.90166107)$$

$$\approx -\ 40.85$$

(b)

$$L(\theta) = f(1100)f(3200)f(3300)f(3500)f(3900).[S(4000)]^{495}$$
$$= \theta^{-1}e^{-1100/\theta}\theta^{-1}e^{-3200/\theta}\theta^{-1}e^{-3300/\theta}\theta^{-1}e^{-3500/\theta}\theta^{-1}e^{-3900/\theta}[e^{-4000/\theta}]^{495}$$
$$= \theta^{-5}e^{-1995000/\theta}$$

$$\Rightarrow l(\theta) = -5\ln\theta - \frac{1995000}{\theta}$$

To get $\hat{\theta}$, set $l'(\theta) = 0$

$$\therefore\ \frac{\partial l}{\partial\theta} = \frac{-5}{\theta} + \frac{1995000}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \frac{1995000}{5} = 399000$$

and $l(\hat{\theta}) = -69.4836$

Q4: 10[2+2+3+3]

(i)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\ \Gamma(4)}, \qquad \theta > 0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta}(1000)^4}{6}, \qquad \theta > 0$$

(ii)

The marginal density at the observed values is

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta)\right]\pi(\theta)d\theta$$

$$f(100,950,450)=\int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$f(100,950,450)=\frac{1000^4}{6}\int_0^\infty \theta^6 e^{-2500\theta}d\theta$$

Let $t=2500\theta$

$$f(100,950,450)=\frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6}e^{-t}\frac{dt}{2500}$$

$$f(100,950,450)=\frac{1000^4}{6(2500)^7}\int_0^\infty t^6 e^{-t}dt$$

$$\therefore f(100,950,450)=\frac{1000^4}{6(2500)^7}\Gamma(7)$$

$$\therefore f(100,950,450)=\frac{1000^4}{6}\frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4)=\int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$=\frac{1000^4}{6}\int_0^\infty \theta^7 e^{-(2500+x_4)\theta}d\theta$$

$$=\frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$=\frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4\,|100,950,450)=\frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e.

$$f(x_4\,|100,950,450)=\frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameter 7 and 2500.

To get posterior density of Θ given X, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})=\frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}}{\frac{1000^4}{6}\ \frac{720}{(2500)^7}}$$

$$= \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(iii) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}|\mathrm{X}=\mathrm{x}) = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4|100,950,450) = \int_0^\infty \frac{1}{\theta}.\ \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}d\theta$$

$$= \frac{(2500)^7}{720}\int_0^\infty \theta^5 e^{-2500\theta}d\theta$$

$$= \frac{2500}{720}\int_0^\infty u^5 e^{-u}du,\ \ u = 2500\theta$$

$$= \frac{2500}{720}\Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4|100,950,450) = 416.67$$

Another solution

$$\because f(x_4|100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density function with parameters $(\alpha,\theta) = (7,2500)$

$$\therefore E(x_4|100,950,450) = \frac{\theta}{\alpha-1} = \frac{2500}{6} = 416,67.$$

(iv) To get the Bühlmann premium,

The hypothetical mean is $\mu(\Theta) = \Theta^{-1}$ where $\Theta \sim gamma(4,1000)$, $\alpha = 4$ and $\beta = 1000$ (the reciprocal of the usual scale parameter).

$\mu = E[\mu(\Theta)] = E(\Theta^{-1}) = \dfrac{\beta}{\alpha-1} = \dfrac{1,000}{3}$ is the expected value of hypothetical mean,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta^{-2}) = \dfrac{\beta^2}{(\alpha-1)(\alpha-2)} = \dfrac{500,000}{3}$ is the expected value of process variance and

$a = Var(\Theta^{-1}) = \dfrac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \dfrac{500,000}{9}$ is the variance of hypothetical mean.

We can also find a as $a = Var(\Theta^{-1}) = \dfrac{500,000}{3} - \left(\dfrac{1,000}{3}\right)^2 = \dfrac{500,000}{9}$

$$\therefore k = \frac{\upsilon}{a} = 3,\ \text{Z}= \frac{n}{n+k} = \frac{3}{6} = \frac{1}{2}.$$

So, the Bühlmann Premium is

$$P_c = Z\overline{X} + (1-Z)\mu$$
$$\therefore P_c = \frac{1}{2}(500) + \frac{1}{2}(\frac{1000}{3}) = 416.67, \quad \text{where } \overline{X} = \frac{100+950+450}{3} = 500,$$

which matches the result of Bayesian premium that introduced in (iii).

Q5: 10[4+6]

(a)

at $p = 0.90, \Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p/r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + \left(\frac{\sigma}{\theta}\right)^2]$$

where σ^2=7500^2, θ=2500

$$\therefore \text{ The expected \# of claims } = 1082.41[1 + \left(\frac{7500}{2500}\right)^2] = 10824.1$$

(b)

(i) The pgf of each X_j is $P_{X_j}(z|\theta) = e^{\theta(z-1)}$

$\Rightarrow$The pgf of S is $P_{S|\Theta}(s|\theta) = e^{n\theta(z-1)}$ which is a Poisson with mean $n\theta$, $f_{S|\Theta}(s|\theta) = \dfrac{(n\theta)^s e^{-n\theta}}{s!}, \quad s = 0,1,2,\ldots$

$$\therefore f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!}\pi(\theta)d\theta, \quad s = 0,1,2,\ldots \quad (1)$$

(ii) The Bayesian Premium is

$\int_0^\infty \mu(\theta)\pi(\theta|\mathrm{x})d\theta$ where $\mu(\theta) = \mathrm{E}(X|\theta) = \theta$

and posterior distribution $\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})$ is given by

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x}) = \frac{\left[\prod_{j=1}^{n} f(x_j|\theta)\right]\pi(\theta)}{\int_0^\infty \left[\prod_{j=1}^{n} f(x_j|\theta)\right]\pi(\theta)d\theta} = \frac{\theta^s e^{-n\theta}\pi(\theta)}{\int_0^\infty \theta^s e^{-n\theta}\pi(\theta)d\theta}, \quad s = \sum_1^n x_j$$

$$\because \mathrm{E}(X_{n+1}|\mathrm{X}=\mathrm{x}) = \frac{\int_0^\infty \theta^{s+1}e^{-n\theta}\pi(\theta)d\theta}{\int_0^\infty \theta^s e^{-n\theta}\pi(\theta)d\theta} = \frac{\frac{(s+1)!}{n^{s+1}}\int_0^\infty \frac{n^{s+1}}{(s+1)!}\theta^{s+1}e^{-n\theta}\pi(\theta)d\theta}{\frac{s!}{n^s}\int_0^\infty \frac{n^s}{s!}\theta^s e^{-n\theta}\pi(\theta)d\theta} \quad (2)$$

By combining Eqs (1) and (2), we get $\mathrm{E}(X_{n+1}|\mathrm{X}=\mathrm{x}) = \frac{s+1}{n}\frac{f_S(s+1)}{f_S(s)} = \frac{1+n\bar{x}}{n}\frac{f_S\left(1+n\bar{x}\right)}{f_S\left(n\bar{x}\right)}, \quad s = \sum_1^n x_j = n\bar{x}.$

(iii) $f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!}\pi(\theta)d\theta, \quad s = 0,1,2,\ldots$

when $\Theta \sim gamma(\alpha,\beta)$, $\pi(\theta) = \frac{\theta^{\alpha-1}\beta^{-\alpha}e^{-\theta/\beta}}{\Gamma(\alpha)}$,

$$f_S(s) = \int_0^\infty \frac{(n\theta)^s e^{-n\theta}}{s!}\frac{\theta^{\alpha-1}\beta^{-\alpha}e^{-\theta/\beta}}{\Gamma(\alpha)}d\theta,$$

$$f_S(s) = \frac{n^s\beta^{-\alpha}}{s!\Gamma(\alpha)}\int_0^\infty \theta^{s+\alpha-1}e^{-(n+\beta^{-1})\theta}d\theta$$

$$\therefore f_S(s) = \frac{n^s\beta^{-\alpha}}{s!\Gamma(\alpha)}\frac{\Gamma(s+\alpha)}{(n+\beta^{-1})^{s+\alpha}} \quad (1)$$

$$f_S(s) = \binom{\alpha+s-1}{s}n^s\beta^{-\alpha}\left(\frac{1}{n+\beta^{-1}}\right)^{\alpha}\left(\frac{1}{n+\beta^{-1}}\right)^{s}$$

$$= \binom{\alpha+s-1}{s}n^s\left(\frac{1}{1+n\beta}\right)^{\alpha}\left(\frac{n\beta}{1+n\beta}\right)^{s}$$

$$\therefore f_S(s) = \binom{r+s-1}{s}\left(\frac{1}{1+\beta^*}\right)^r \left(\frac{\beta^*}{1+\beta^*}\right)^s \quad (2)$$

where $\beta^* = n\beta$ and $r = \alpha$, which is a negative binomial distribution.