King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2-1446H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [4+6]

(a) One hundred observed claims in 2022 were arranged as follows: 40 were between 0 and 300, 10 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 10 were between 500 and 600, and the remaining 30 were above 600. For the next three years, all claims are inflated by 10% per year. Based on the empirical distribution from 2022, determine a range for the probability that a claim exceeds 500 in 2025.

(b) An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses: 100(0.4), 500(0.2), 1000(0.2), 2500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for the excess loss, and left censored and shifted variables, where the deductible is set at 750. Also, calculate the probability mass function and the expected value of the limited loss variable (right censored variable) with a limit of 750.

Q2: [2+4]

(a) Consider a frailty model with frailty random variable Λ, such that $a(x)=\dfrac{1}{x+1}, \quad x>0.$

Find the conditional survival function of X.

(b) Let X have a Pareto distribution, where $F(x)=1-\left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Identify the names of these distributions.

Q3: [6]

A random sample of size 5 is taken from a Weibull distribution with $\tau = 2$. Two of the sample observations are known to exceed 50 and the three remaining observations are 20, 30, and 45. Determine the maximum likelihood estimate of θ. Also, find the value of the log-likelihood function.

Hint: For Weibull distribution, $F(x) = 1 - e^{-(x/\theta)^{\tau}}$

Q4: [5+5]

(a) Risk 1 produces claims of amounts 100, 1,000, and 20,000 with probabilities 0.5, 0.3, and 0.2, respectively. For risk 2, the probabilities are 0.7, 0.2 and 0.1. Risk 2 is twice as likely as risk 1 of being observed. A claim of 100 is observed, but the observed risk is unknown. Determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk.

(b) The amount of a claim X has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$, β here is the reciprocal of the usual scale parameter of gamma distribution. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(i) The probability density functions for X, and risk parameter Θ.

(ii) The predictive distribution of the fourth claim and the posterior distribution of Θ.

(iii) The Bayesian premium.

Q5: [4+4]

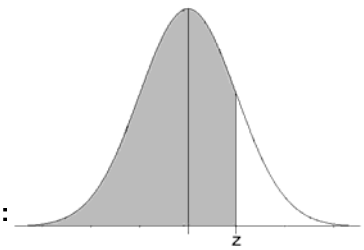
(a) The average claim size for a group of insureds is 2500, with a standard deviation of 7500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that the number of claims from m_j policies is N_j in year j for a group policyholder with risk parameter Θ has a Poisson distribution with mean $m_j\Theta$, that is, for $j = 1, \ldots, n$,

$$\Pr(N_j = x | \Theta = \theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!}, \ x = 0, 1, 2, \ldots,$$

where Θ has a gamma distribution with parameters α and β. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the m_{n+1} policies.

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+6]

(a)

The amount in 2022	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	40	10	5	5	0	10	30

For the next three years, all claims are inflated by 10% per year

In 2023 $\rightarrow$ 1.1 X, in 2024 $\rightarrow$ 1.21 X and in 2025 $\rightarrow$ 1.331 X

where X is the random variable of the claim in 2023 and Y=1.331 X is the random variable of the claim in 2025. $\Pr(Y > 500) = \Pr(X > 500/1.331) = \Pr(X > 376)$

From given data, $\Pr(X > 350)$ =50/100=0.50 and $\Pr(X > 400)$ =45/100=0.45

$\therefore$ 0.45<$\Pr(Y > 500) < 0.50$

(b)

The probability of exceeding the deductible is

$$Pr(X > 750) = 1 - F(750), d = 750$$
$$= 0.2 + 0.1 + 0.1 = 0.4$$

For the excess loss variable $Y^p = X - d$, we have

$x_j - d$: 250 1750 9250

$\frac{p(x_j)}{1-F(d)}$: 0.2/0.4 0.1/0.4 0.1/0.4

0.5 0.25 0.25

The expected value for the excess loss variable is defined as

$$e_X(d) = E(Y^p) = E(X - d | X > d)$$

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2{,}875$$

For the left censored and shifted variable $Y^L = (X - d)_+$

$(x_j - d)_+$: 0 250 1750 9250

$p(x_j)$: 0.6 0.2 0.1 0.1

The expected value for the left censored and shifted variable is

$E(Y^L) = E[(X-d)_+]$, where $Y^L = \begin{cases} 0, & X \le d \\ X-d, & X > d \end{cases}$

$\therefore E(Y^L) = E[(X-d)_+]$

$= 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$

$= 1{,}150$

For the limited loss variable (right censored variable)

$x_j \wedge d$:	100	500	750
$p(x_j)$:	0.4	0.2	0.4

The expected value for the limited loss variable (right censored variable) is

$E(Y) = E(X \wedge d)$, where $Y = \begin{cases} X, & X < d \\ d, & X \ge d \end{cases}$

$\therefore E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$

$= 440$

Note that: $E(X) = E[(X-d)_+] + E(X \wedge d)$

$= 1{,}150 + 440$

$= 1{,}590$

Q2: [2+4]

(a)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\ dt$$

$$= \int_0^x \frac{dt}{1+t} = ln(1+x)$$

Thus,

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$

$$= e^{-\lambda\, ln(1+x)}$$

$$= \frac{1}{(1+x)^\lambda}$$

(b)

For Pareto distribution with parameters α, θ $F_X(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$

For $\tau > 0,\ F_Y(y) = F_X(y^\tau)$

$$\therefore F_Y(y) = 1 - \left(\frac{\theta}{y^\tau + \theta}\right)^\alpha$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1 + (y/\theta^{1/\tau})^\tau}\right)^\alpha$$

which is the Burr distribution with three parameters $\alpha, \theta^{1/\tau}, \tau$

For $\tau = -1,\ F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{y^{-1} + \theta}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{y^{-1} + \theta}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{y}{y + \theta^{-1}}\right)^{\alpha}$$

which is the inverse Pareto distribution with parameters α, θ^{-1}

For negative $\tau,\quad F_Y(y) = 1 - F_X(y^{-\tau})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + (\theta^{-1/\tau})^{\tau}}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{\left(y/\theta^{-1/\tau}\right)^{\tau}}{1 + \left(y/\theta^{-1/\tau}\right)^{\tau}}\right)^{\alpha}$$

which is the inverse Burr distribution with three parameters $\alpha, \theta^{-1/\tau}, \tau$

Q3: [6]

For Weibull distribution

$$F(x) = 1 - e^{-(x/\theta)^2}, f(x) = \frac{2x}{\theta^2} e^{-(x/\theta)^2}$$

The likelihood function is

$$L(\theta) = f(20)f(30)f(45)[1 - F(50)]^2$$

$$\therefore L(\theta) = \frac{40}{\theta^2} e^{-(20/\theta)^2} \frac{60}{\theta^2} e^{-(30/\theta)^2} \frac{90}{\theta^2} e^{-(45/\theta)^2} \left[e^{-(50/\theta)^2}\right]^2$$

$$= 216{,}000\theta^{-6} e^{-8325/\theta^2}$$

$$\therefore l(\theta) = ln\, 216{,}000 - 6\, ln\, \theta - 8325\theta^{-2}$$

which is known as log-likelihood function, to get $\hat{\theta}$ Set $l'(\theta) = 0$

$$\Rightarrow -6\theta^{-1} + 2(8325)\theta^{-3} = 0$$

$$\therefore 6\theta^2 = 16650$$

$\therefore \hat{\theta} = \sqrt{\frac{16650}{6}} \simeq 52.68$

$\therefore l\left(\hat{\theta}\right) \simeq -14.5$

Q4: [5+5]

(a)

The required calculations are given in the following table.

Risk	100	1,000	20,000	$\mu(\Theta)$	$v(\Theta)$	$\Pr(\Theta=\theta)$
1	0.5	0.3	0.2	4,350	61,382,50(	1/3
2	0.7	0.2	0.1	2,270	35,054,10(	2/3

To determine the Bühlmann credibility estimate of the expected value of the second claim amount from the same risk, we should find the following quantities.

$\mu = E[\mu(\Theta)]$

$=(1/3)(4350)+(2/3)(2270)=2{,}963.333,$

$v = E[v(\Theta)]$

$=(1/3)(61{,}382{,}500)+(2/3)(35{,}054{,}100)=43{,}830{,}233.333,$

$a = \text{var}[\mu(\Theta)]$

$=(1/3)(4350)^2+(2/3)(2270)^2 - 2{,}963.333^2 = 961{,}424.1978,$

$k = \frac{v}{a}$

$= 45.58886,$

$Z = \frac{n}{n+k}$

$= \frac{1}{1+45.58886}$

$\therefore$ Z $\approx 0.0215.$

The Bühlmann estimate is

$E(X_2|100) = P_c = Z\overline{X} + (1-Z)\mu$

$= 0.0215(100) + (1-0.0215)(2{,}963.333)$

$= 2{,}901.77.$

where $X_1 = 100.$

(b)

(i)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\,\Gamma(4)}, \qquad \theta>0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta}(1000)^4}{6}, \qquad \theta>0$$

(ii)

The marginal density at the observed values is

$$f_X(x) = \int\left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta)\right]\pi(\theta)d\theta$$

$$f(100,950,450) = \int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$f(100,950,450) = \frac{1000^4}{6}\int_0^\infty \theta^6 e^{-2500\theta}d\theta$$

Let $t = 2500\theta$

$$f(100,950,450) = \frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6}e^{-t}\frac{dt}{2500}$$

$$f(100,950,450) = \frac{1000^4}{6(2500)^7}\int_0^\infty t^6 e^{-t}dt$$

$$\therefore f(100,950,450) = \frac{1000^4}{6(2500)^7}\Gamma(7)$$

$$\therefore f(100,950,450) = \frac{1000^4}{6}\frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4) = \int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$= \frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$= \frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4|100,950,450) = \frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e.

$$f(x_4|100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameter 7 and 2500.

To get posterior density of Θ given X, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x}) = \frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \dfrac{1000^4}{6}\theta^3 e^{-1000\theta}}{\dfrac{1000^4}{6}\ \dfrac{720}{(2500)^7}}$$

$$= \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(iii) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}|\mathrm{X{=}x}) = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4|100,950,450) = \int_0^\infty \frac{1}{\theta}\cdot\ \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}d\theta$$

$$= \frac{(2500)^7}{720}\int_0^\infty \theta^5 e^{-2500\theta}d\theta$$

$$= \frac{2500}{720}\int_0^\infty u^5 e^{-u}du,\ \ u = 2500\theta$$

$$= \frac{2500}{720}\Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4|100,950,450) = 416.67$$

Q5: [4+4]

(a)

at $p = 0.90, \Phi(y_p) = (1+p)/2 = 0.95$

$\Rightarrow y_p = 1.645$ (by using SND table)

$\Rightarrow \lambda_0 = (y_p/r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\frac{\sigma}{\theta})^2]$$

where σ^2=7500^2, θ=2500

$\therefore$ The expected # of claims $= 1082.41[1 + \left(\frac{7500}{2500}\right)^2]$

$= 10824.1$

(b)

Let $X_j = N_j / m_j$ be the average of claims per individual in year j.

$\because N_j | \Theta$ has a Poisson distribution with mean $m_j\Theta$, and variance $m_j\Theta$.

$\therefore E(X_j | \Theta) = E\left(\frac{N_j}{m_j} \middle| \Theta\right) = \frac{m_j \Theta}{m_j} = \Theta = \mu(\Theta)$ hypothetical mean

and

$$Var(X_j | \Theta) = Var\left(\frac{N_j}{m_j} \middle| \Theta = \theta\right)$$

$$= \frac{1}{m_j^2} Var(N_j | \Theta) = \frac{m_j \Theta}{m_j^2}$$

$$= \frac{\Theta}{m_j} = \frac{v(\Theta)}{m_j}$$

$\therefore v(\Theta) = \Theta$ process variance

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical mean (collective premium), where

$\Theta \sim gamma(\alpha, \beta)$,

$v = E[v(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$$\therefore k = \frac{v}{a} = \frac{1}{\beta}, Z = \frac{m}{m+k} = \frac{m\beta}{m\beta+1}.$$

So, the Bühlmann-Straub estimate for one policyholder is

$$P_c = \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu$$

$$= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X} = \text{m}^{-1}\sum_{j=1}^{n} m_j X_j$$

For m_{n+1} policies in year $n+1$, the estimate is $m_{n+1}P_c$.