King Saud University
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Department of Mathematics

Final Exam, S1-1445H
ACTU 475
Credibility Theory and Loss Distributions.
Time: 3 hours - Marks: 40

Answer the following questions.

(Note that SND Table is attached in page 3)

Q1: [4+6]

(a) One hundred observed claims in 1995 were arranged as follows: 42 were between 0 and 300, 3 were between 300 and 350, 5 were between 350 and 400, 5 were between 400 and 450, 0 were between 450 and 500, 5 were between 500 and 600, and the remaining 40 were above 600. For the next three years, all claims are inflated by 10% per year. Based on the empirical distribution from 1995, determine a range for the probability that a claim exceeds 500 in 1998.

(b) An automobile insurance policy with no coverage modifications has the following possible losses, with probabilities in parentheses: 100(0.4), 500(0.2), 1000(0.2), 2500(0.1), and 10,000(0.1). Determine the probability mass functions and expected values for the excess loss, and left censored and shifted variables, where the deductible is set at 750. Also, calculate the probability mass function and the expected value of the limited loss variable (right censored variable) with a limit of 750.

Q2: [2+4]

(a) Consider a frailty model with frailty random variable Λ, such that $a(x)=\dfrac{1}{x+1}, \quad x>0.$

Find the conditional survival function of X.

(b) Let X have a Pareto distribution, where $F(x)=1-\left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$. Determine the cdf of the inverse, transformed, and inverse transformed distributions. Identify the names of these distributions.

Q3: [6]

A random sample of size 5 is taken from a Weibull distribution with $\tau = 2$. Two of the sample observations are known to exceed 50 and the three remaining observations are 20, 30, and 45. Determine the maximum likelihood estimate of θ. Also, find the value of the log-likelihood function.

Hint: For Weibull distribution, $F(x) = 1 - e^{-(x/\theta)^{\tau}}$

Q4: [5+5]

(a) Risk 1 produces claims of amounts 100, 1,000, and 20,000 with probabilities 0.5, 0.3, and 0.2, respectively. For risk 2, the probabilities are 0.7, 0.2 and 0.1. Risk 1 is twice as likely as risk 2 of being observed. A claim of 100 is observed, but the observed risk is unknown. Determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk.

(b) If the amount of a claim has an exponential distribution with mean $\frac{1}{\theta}$ where θ is the risk parameter, and $\Theta \sim gamma(\alpha, \beta)$ where β is the reciprocal of the usual scale parameter. Determine the Bühlmann credibility premium.

Q5: [4+4]

(a) The average claim size for a group of insureds is 1500, with a standard deviation of 7500. Assume that claim counts have the Poisson distribution. Determine the expected number of claims so that the total loss will be within 5% of the expected total loss with probability 0.90.

(b) Suppose that the number of claims from m_j policies is N_j in year j for a group policyholder with risk parameter Θ has a Poisson distribution with mean $m_j\Theta$, that is, for $j = 1,\ldots,n$,

$$\Pr(N_j = x \mid \Theta = \theta) = \frac{(m_j\theta)^x e^{-m_j\theta}}{x!},\ x = 0,1,2,\ldots,$$

where Θ has a gamma distribution with parameters α and β. Determine the Bühlmann- Straub estimate of the expected number of claims in year $n+1$ for the m_{n+1} policies.

Standard Normal Cumulative Probability Table

Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

The Model Answer

Q1: [4+6]

(a)

The amount in 1995	0-300	300-350	350-400	400-450	450-500	500-600	600-
# of claims	42	3	5	5	0	5	40

For the next three years, all claims are inflated by 10% per year

In 1996 → 1.1 X, in 1997→ 1.21 X and in 1998 → 1.331 X

where X is the random variable of the claim in 1995 and Y=1.331 X is the random variable of the claim in 1998.

$Pr(Y > 500) = Pr(X > 500/1.331) = Pr(X > 376)$

From given data, $Pr(X > 350)$ =55/100=0.55 and $Pr(X > 400)$ =50/100=0.50

$\therefore$ 0.50<Pr($Y > 500) < 0.55$

(b)

The probability of exceeding the deductible is

$$Pr(X > 750) = 1 - F(750), d = 750$$
$$= 0.2 + 0.1 + 0.1 = 0.4$$

For the excess loss variable $Y^p = X - d$, we have

$$x_j - d: \quad 250 \quad 1750 \quad 9250$$
$$\frac{p(x_j)}{1 - F(d)}: \quad 0.2/0.4 \quad 0.1/0.4 \quad 0.1/0.4$$
$$0.5 \quad 0.25 \quad 0.25$$

The expected value for the excess loss variable is defined as

$$e_X(d) = E(Y^p) = E(X - d|X > d)$$

$$\therefore e_X(d) = 250(0.5) + 1750(0.25) + 9250(0.25)$$
$$= 2{,}875$$

For the left censored and shifted variable $Y^L = (X - d)_+$

$(x_j - d)_+$:	0	250	1750	9250
$p(x_j)$:	0.6	0.2	0.1	0.1

The expected value for the left censored and shifted variable is

$$E(Y^L) = E[(X-d)_+], \text{ where } Y^L = \begin{cases} 0, & X \le d \\ X-d, & X > d \end{cases}$$

$$\therefore E(Y^L) = E[(X-d)_+]$$
$$= 0(0.6) + 250(0.2) + 1750(0.1) + 9250(0.1)$$
$$= 1{,}150$$

For the limited loss variable (right censored variable)

$x_j \wedge d$:	100	500	750
$p(x_j)$:	0.4	0.2	0.4

The expected value for the limited loss variable (right censored variable) is

$$E(Y) = E(X \wedge d), \text{ where } Y = \begin{cases} X, X < d \\ d, X \ge d \end{cases}$$

$$\therefore E(X \wedge d) = 100(0.4) + 500(0.2) + 750(0.4)$$
$$= 440$$

Not that: $E(X) = E[(X-d)_+] + E(X \wedge d)$
$$= 1{,}150 + 440$$
$$= 1{,}590$$

Q2: [2+4]

(a)

We first find $A(x)$

$$A(x) = \int_0^x a(t)\ dt$$
$$= \int_0^x \frac{dt}{1+t} = ln(1+x)$$

Thus,

$$S_{X|\Lambda}(x|\lambda) = e^{-\lambda A(x)}$$
$$= e^{-\lambda\, ln(1+x)}$$
$$= \frac{1}{(1+x)^{\lambda}}$$

(b)

For Pareto distribution with parameters α, θ $F_X(x) = 1 - \left(\dfrac{\theta}{x+\theta}\right)^{\alpha}$

For $\tau > 0,\ F_Y(y) = F_X(y^{\tau})$

$$\therefore F_Y(y) = 1 - \left(\frac{\theta}{y^{\tau}+\theta}\right)^{\alpha}$$

$$\therefore F_Y(y) = 1 - \left(\frac{1}{1+(y/\theta^{1/\tau})^{\tau}}\right)^{\alpha}$$

Which is the Burr distribution with three parameters $\alpha, \theta^{1/\tau}, \tau$

For $\tau = -1,\ F_Y(y) = 1 - F_X(y^{-1})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{y^{-1}+\theta}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{y^{-1}+\theta}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{y}{y+\theta^{-1}}\right)^{\alpha}$$

which is the inverse Pareto distribution with parameters α, θ^{-1}

For negative τ, $F_Y(y) = 1 - F_X(y^{-\tau})$

$$\therefore F_Y(y) = 1 - \left[1 - \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}\right]$$

$$= \left(\frac{\theta}{\theta + y^{-\tau}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + \theta^{-1}}\right)^{\alpha}$$

$$= \left(\frac{y^{\tau}}{y^{\tau} + (\theta^{-1/\tau})^{\tau}}\right)^{\alpha}$$

$$\therefore F_Y(y) = \left(\frac{(y/\theta^{-1/\tau})^{\tau}}{1+(y/\theta^{-1/\tau})^{\tau}}\right)^{\alpha}$$

which is the inverse Burr distribution with three parameters $\alpha, \theta^{-1/\tau}, \tau$

Q3: [6]

For Weibull distribution

$$F(x) = 1 - e^{-(x/\theta)^2}, f(x) = \frac{2x}{\theta^2} e^{-(x/\theta)^2}$$

The likelihood function is

$$L(\theta) = f(20)f(30)f(45)[1 - F(50)]^2$$
$$\therefore L(\theta) = \frac{40}{\theta^2} e^{-(20/\theta)^2} \frac{60}{\theta^2} e^{-(30/\theta)^2} \frac{90}{\theta^2} e^{-(45/\theta)^2} \left[e^{-(50/\theta)^2}\right]^2$$
$$= 216{,}000\theta^{-6} e^{-8325/\theta^2}$$
$$\therefore l(\theta) = ln\, 216{,}000 - 6\, ln\, \theta - 8325\theta^{-2}$$

which is known as log-likelihood function, to get $\hat{\theta}$ Set $l'(\theta) = 0$

$$\Rightarrow -6\theta^{-1} + 2(8325)\theta^{-3} = 0$$
$$\therefore 6\theta^2 = 16650$$
$$\therefore \hat{\theta} = \sqrt{\frac{16650}{6}} \simeq 52.68$$

$$\therefore l(\theta) \simeq -14.5$$

Q4: [5+5]

(a)

The required calculations are given in the following table.

Risk	100	1,000	20,000	$\mu(\Theta)$	$v(\Theta)$	$\Pr(\Theta = \theta)$
1	0.5	0.3	0.2	4,350	61,382,50[illegible]	2/3
2	0.7	0.2	0.1	2,270	35,054,10[illegible]	1/3

To determine the Bayesian credibility estimate of the expected value of the second claim amount from the same risk, we do the following.

Clearly, $\pi(\theta = 1) = \frac{2}{3}$, $\pi(\theta = 2) = \frac{1}{3}$

The marginal probability is $f_X(x) = \sum_{\theta} f(x|\theta)\pi(\theta)$

$$f(100) = f(100|1)\pi(1) + f(100|2)\pi(2)$$
$$= 0.5\left(\frac{2}{3}\right) + 0.7\left(\frac{1}{3}\right) = \frac{17}{30}$$

The posterior probabilities are given by

$$\pi(1|100) = \frac{f(100|1)\pi(1)}{f(100)} = \frac{10}{17} \text{ and } \pi(2|100) = 1 - \frac{10}{17} = \frac{7}{17}$$

The hypothetical means are

$$\mu(1) = 4350,\ \mu(2) = 2270$$

The expected next value through Bayesian premium is

$$E(X_2|100) = \pi(1|100)\mu(1) + \pi(2|100)\mu(2)$$
$$= 3493.53$$

where $X_1 = 100$.

(b)

$\mu(\theta) = E(X_j|\Theta = \theta) = \frac{1}{\theta}$ (1) hypothetical mean

$v(\theta) = Var(X_j|\Theta = \theta) = \frac{1}{\theta^2}$ (2) process variance

The expected value of hypoth. mean (collective premium) is

$$\mu = E(\Theta^{-1}) = \int_0^{\infty} \frac{1}{\theta} . \frac{(\theta\beta)^{\alpha} e^{-\beta\theta}}{\theta\Gamma(\alpha)} d\theta$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} \theta^{\alpha-2} e^{-\beta\theta} d\theta$$

$$= \frac{\beta}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha-2} e^{-u} du,\ u = \beta\theta$$

$$\therefore \mu = E(\Theta^{-1}) = \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha - 1) = \frac{\beta}{\alpha - 1} \quad (3)$$

The expected value of the process variance is given by

$$v = E[\Theta^{-2}] = \int_0^{\infty} \frac{1}{\theta^2} . \frac{(\beta\theta)^{\alpha} e^{-\beta\theta}}{\theta\Gamma(\alpha)} d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-3} e^{-\beta\theta} d\theta$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-3} e^{-u} du, \qquad u = \beta\theta$$

$$\therefore v = E[\Theta^{-2}] == \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha - 2) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} \quad (4)$$

The variance of hypoth. mean is

$$a = Var(\Theta^{-1}) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}$$

$$= \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (5)$$

$$(4),(5) \Rightarrow k = \frac{v}{a} = \alpha - 1 \quad (6)$$

The Bühlmann credibility factor is

$$Z = \frac{n}{n+k} = \frac{n}{n+\alpha-1} \quad (7)$$

The Bühlmann credibility premium is given by

$$P_c = Z\bar{X} + (1-Z)\mu$$

$(3),(7) \Rightarrow$

$$P_c = \frac{n}{n+\alpha-1}\bar{X} + \left(1 - \frac{n}{n+\alpha-1}\right)\frac{\beta}{\alpha-1}$$

$$\therefore P_c = \frac{n}{n+\alpha-1}\bar{X} + \frac{\alpha-1}{n+\alpha-1}\frac{\beta}{\alpha-1}$$

Q5: [4+4]

(a)

at $p = 0.90, \Phi(y_p) = (1+p)/2 = 0.95$
$\Rightarrow y_p = 1.645$ (by using SND table)
$\Rightarrow \lambda_0 = (y_p/r)^2 = (1.645/0.05)^2 = 1082.41$

To get the expected number of claims, use the following formula:

$$n\lambda = \lambda_0[1 + (\frac{\sigma}{\theta})^2]$$

where σ^2=7500^2, θ=1500

$\therefore$ The expected # of claims $= 1082.41[1 + \left(\frac{7500}{1500}\right)^2]$
$= 28142.66$

(b)

Let $X_j = N_j / m_j$ be the average of claims per individual in year j.

$\because N_j | \Theta$ has a Poisson distribution with mean $m_j \Theta$,

$$E(X_j|\Theta) = E\left(\frac{N_j}{m_j}\middle| \Theta\right) = \frac{m_j \Theta}{m_j} = \Theta = \mu(\Theta)$$

and

$$\begin{aligned} Var(X_j|\Theta) &= Var\left(\frac{N_j}{m_j}\middle| \Theta = \theta\right) \\ &= \frac{1}{m_j^2} Var(N_j|\Theta) = \frac{m_j \Theta}{m_j^2} \\ &= \frac{\Theta}{m_j} = \frac{v(\Theta)}{m_j} \end{aligned}$$

$\Rightarrow$

$\mu = E[\mu(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of hypothetical means, where $\Theta \sim gamma(\alpha, \beta)$,

$v = E[v(\Theta)] = E(\Theta) = \alpha\beta$ is the expected value of process variance and $a = Var(\Theta) = \alpha\beta^2$ is the variance of hypothetical means.

$\therefore k = \frac{v}{a} = \frac{1}{\beta}, Z = \frac{m}{m+k} = \frac{m\beta}{m\beta + 1}$.

So, the Bühlmann-Straub estimate for one policyholder is

$$\begin{aligned} P_c &= \frac{m\beta}{m\beta+1}\overline{X} + \left(1 - \frac{m\beta}{m\beta+1}\right)\mu \\ &= \frac{m\beta}{m\beta+1}\overline{X} + \frac{1}{m\beta+1}\alpha\beta \text{ where } \overline{X} = \mathrm{m}^{-1}\sum_{j=1}^{n} m_j X_j \end{aligned}$$

For year $n+1$, the estimate is $m_{n+1} P_c$.