لا يكتب في هذا الهامش

Q:

Let $x_1 = 1$, $x_{n+1} = \frac{4x_n + 2}{x_n + 3}$ for all $n \in \mathbb{N}$

Show that the sequence (x_n) converges.

Solution:

We will change the look of x_{n+1} as follows:

$$x_{n+1} = \frac{4x_n + 12 - 12 + 2}{x_n + 3} = \frac{4(x_n+3)}{x_n+3} - \frac{10}{x_n+3}$$

(This will help!) $\quad x_{n+1} = 4 - \frac{10}{x_n + 3}$

$$x_2 = \frac{4x_1 + 2}{x_1 + 3} = \frac{4(1)+2}{1+3} = \frac{6}{4} = 1.5 \Rightarrow x_1 < x_2$$

We will prove (x_n) increases by Mathematical induction

Assume " $x_{n+1} \geq x_n$ " is true statment.

Now $x_{n+2} - x_{n+1} = 4 - \frac{10}{x_{n+1}+3} - 4 + \frac{10}{x_n + 3}$

$$= \frac{10}{x_n+3} - \frac{10}{x_{n+1}+3}$$

$$= \frac{10x_{n+1} + 30 - 10x_n - 30}{(x_n+3)(x_{n+1}+3)}$$

$$= \frac{10(x_{n+1} - x_n)}{(x_n+3)(x_{n+1}+3)} \geq 0$$

So $x_{n+2} \geq x_{n+1} \Rightarrow (x_n)$ is increasing.

Now we will prove that $(x_n) \leq M \ \forall n \in \mathbb{N}$, $M \in \mathbb{R}^+$

$x_1 = 1 < 10$. By Mathematical induction:

Assume $x_n \leq 10 \Rightarrow 3 + x_n \leq 13 \Rightarrow \frac{1}{x_n+3} \geq \frac{1}{13}$

$$\Rightarrow \frac{-10}{x_n+3} \leq \frac{-10}{13} \Rightarrow 4 - \frac{10}{x_n+3} \leq 4 - \frac{10}{13}$$

$\Rightarrow x_{n+1} \leq \frac{42}{13} < 10 = M \Rightarrow (x_n)$ is bdd above by 10

So (x_n) is Convergent