9 Normal Subgroups and Factor Groups

Exercise 1: Let $H = \{(1), (12)\}$. Is H normal in S_3?

Solution: No, $(13)(12)(13)^{-1} = (23)$ is not in H.

Exercise 2: Prove that A_n is normal in S_n.

Solution: Note that $|S_n : A_n| = 2$, so A_n has index 2 in S_n. Therefore A_n is normal in S_n.

Exercise 5: Show that if G is the internal direct product of $H_1, H_2, \ldots, H_n$ and $i \neq j$ with $1 \leq i \leq n$, $1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$.

Solution: Say $i < j$and $h \in H_i \cap H_j$. Then $h \in H_1 H_2 \cdots H_{i-1} H_i \cap H_j = \{e\}$. (By the definition of internal direct product.)

Exercise 6: Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$?

Solution: No. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then A is in H and B is in $GL(2, \mathbb{R})$ but BAB^{-1} is not in H.

Exercise 11: Prove that a factor group of a cyclic group is cyclic.

Solution: Let $G = \langle a \rangle$. Then $G/H = \langle aH \rangle$.

Exercise 12: Prove that a factor group of an Abelian group is Abelian.

Solution: Let G be Abelian and H be a normal subgroup of G. For any $aH, bH \in G/H$, we have $(aH)(bH) = abH = baH = (bH)(aH)$. Therefore G/H is Abelian.

Exercise 13: Let H be a normal subgroup of a finite group G and let a be an element of G. Complete the following statement: The order of the element aH in the factor group G/H is the smallest positive integer n such that a^n is ___.

Solution: In H.

Exercise 14: What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?

Solution: Need to find the smallest positive integer n such that $n \cdot 14 \in \langle 8 \rangle = \{0, 8, 16\}$ in $\mathbb{Z}_{24}$. Computing: $14 \cdot 1 = 14$, $14 \cdot 2 = 28 \equiv 4 \pmod{24}$,
$14 \cdot 3 \equiv 18 \pmod{24}$, $14 \cdot 4 \equiv 8 \pmod{24}$. Since $8 \in \langle 8 \rangle$, the order is 4.

Exercise 20: Prove that $U(40)/U_8(40)$ is **NOT** cyclic but $U(40)/U_5(40)$ is cyclic.

Solution: First note that both quotient groups have order 4. For the first part, let $nU_8(40) \in U(40)/U_8(40)$. Then n is odd so $n = 2k+1$ so $n^2 = 4k(k+1)+1 \equiv 1(mod 8)$, this is because $k(k+1)$ is even. Now this means $U(40)/U_8(40)$ has no element of order 4. Therefore its not cyclic. For the second part, $U(40)/U_5(40)$ has order $4 = |3U_5(40)|$. So $U(40)/U_5(40) = \langle 3U_5(40) \rangle$ is cyclic.

Exercise 28: Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $H = \{(0,0),(2,0),(0,2),(2,2)\}$, and $K = \langle (1,2) \rangle$. Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$? Is G/K isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

Solution: Check elements orders to conclude $G/H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; $G/K \cong \mathbb{Z}_4$.

Exercise 30: Express $U(165)$ as an internal direct product of proper subgroups in four different ways.

Solution: Looking for relatively prime numbers that multiply to 165, we have

$$U(165) = U_{15}(165) \times U_{11}(165) = U_{33}(165) \times U_5(165) = U_{55}(165) \times U_3(165).$$

We can also write $U(165) = U_3(165) \times U_5(165) \times U_{11}(165)$ as a fourth way.

Exercise 37: Let G be a finite group and let H be a normal subgroup of G. Prove that for any $g \in G$, $|gH|$ divides $|g|$.

Solution: Say $|g| = n$. Then $(gH)^n = g^n H = eH = H$. By Corollary 2 to Theorem 4.1, the order of gH divides $n = |g|$.

Exercise 39: Let H be a subgroup of a group G with the property that for all a and b in G, $aHbH = abH$. Prove that H is a normal subgroup of G.

Solution: Let x belong to G. Then $xHx^{-1}H = xx^{-1}H = H$, so $xHx^{-1} \subseteq H$.

Exercise 44: Verify that the mapping defined at the end of the proof of Theorem 9.6 is an isomorphism.

Solution: Done in class.

Exercise 47: Let H and K be subgroups of a group G. If $|H| = 63$ and $|K| = 45$, prove that $H \cap K$ is Abelian. Generalize.

Solution: By Lagrange, $|H \cap K|$ divides both 63 and 45. If $|H \cap K| = 9$, then $H \cap K$ is Abelian by Theorem 9.7. If $|H \cap K| = 3$, then $H \cap K$ is cyclic by the Corollary of Theorem 7.1. If $|H \cap K| = 1$, then $H \cap K = \{e\}$.

Generalization: If p is a prime and $|H| = p^2m$ and $|K| = p^2n$ where $\gcd(m, n) = 1$, then $|H \cap K| = 1, p,$ or p^2. So by Corollary 3 of Theorem 7.1 and Theorem 9.7, $H \cap K$ is Abelian.

Exercise 50: If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

Solution: By Lagrange's Theorem, $|Z(G)|$ divides $|G| = pq$. So $|Z(G)| \in \{1, p, q, pq\}$. If $|Z(G)| = p$ or $|Z(G)| = q$, then $|G/Z(G)| = q$ or $|G/Z(G)| = p$ respectively, both prime. By the G/Z Theorem (Theorem 9.3), $G/Z(G)$ cyclic implies G is Abelian, so $Z(G) = G$, which contradicts $|Z(G)| = p$ or q. Therefore $|Z(G)| = 1$ or pq.

Exercise 55: In D_4, let $K = \{R_0, D\}$ and let $L = \{R_0, D, D', R_{180}\}$. Show that $K \triangleleft L \triangleleft D_4$, but that K is not normal in D_4. (Normality is not transitive.)

Solution: We know that K is normal in L (since $|L : K| = 2$) and L is normal in D_4 (since $|D_4 : L| = 2$). But $VK = \{V, R_{270}\}$, whereas $KV = \{V, R_{90}\}$. So K is not normal in D_4. We Did another example in class using A_4.

Exercise 64: Let G be a group and let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy \mid x, y \in G\}$.

a. Prove that G' is normal in G.
b. Prove that G/G' is Abelian.
c. If G/N is Abelian, prove that $G' \leq N$.
d. Prove that if H is a subgroup of G and $G' \leq H$, then H is normal in G.

Solution: (a) For any $g \in G$ and $s = x^{-1}y^{-1}xy \in S$, we have
$gsg^{-1} = g(x^{-1}y^{-1}xy)g^{-1} = (gx^{-1}g^{-1})(gy^{-1}g^{-1})(gyg^{-1})(gxg^{-1}) = (gxg^{-1})^{-1}(gyg^{-1})^{-1}(gyg^{-1})(gxg^{-1}) \in S.$
Since conjugation **preserves products**, $gG'g^{-1} = G'$, so G' is normal.

(b) For any $aG', bG' \in G/G'$, we have $(aG')(bG') = abG'$ and $(bG')(aG') = baG'$.
Now $a^{-1}b^{-1}ab \in G'$, so $abG' = baG'$. Thus G/G' is Abelian.

(c) If G/N is Abelian, then for all $x, y \in G$, $xNyN = yNxN$, which gives $xyN = yxN$.
Therefore $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$. Since G' is generated by such elements,
$G' \leq N$.

(d) If $G' \leq H$, then for all $g \in G$ and $h \in H$, $ghg^{-1}h^{-1} \in G' \subset H$, so $ghg^{-1} \in H$.
Thus H is normal in G.

Exercise 70: Prove that A_4 is the only subgroup of S_4 of order 12.

Solution: Let H be a distinct subgroup of S_4 of order 12. Then $|S_4 : H| = 24/12 = 2$, so by Exercise 9, H is normal in S_4 and $HA_4 = S_4$. Since A_4 also normal $H \cap A_4$ is normal and $|H \cap A_4| = |H||A_4|/24 = 6$. But, as we showed in class, there is no subgroup of A_4 with order 6.