Linear Transformations

- Introduction to Linear Transformations
- The Kernel and Range of a Linear Transformation
- Matrices for Linear Transformations
- Transition Matrices and Linear Transformations

Introduction to Linear Transformations

- A function T that maps a vector space V into a vector space W:

$$T : V \xrightarrow{\text{mapping}} W, \qquad V, W : \text{vector spaces}$$

V: the domain of T $\qquad$ W: the codomain of T

- Image of **v** under T:

If **v** is a vector in V and **w** is a vector in W such that

$$T(\mathbf{v}) = \mathbf{w},$$

then **w** is called the image of **v** under T

(For each **v**, there is only one **w**)

- The range of T:

The set of all images of vectors in V (see the figure on the next slide)

Example: A function from R^2 into R^2

$$T : R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2) (b) Find the preimage of **w**=(-1,11)

Solution:

(a) $\mathbf{v} = (-1,\ 2)$

$$\Rightarrow T(\mathbf{v}) = T(-1,\ 2) = (-1-2,\ \ -1+2(2)) = (-3,\ 3)$$

(b) $T(\mathbf{v}) = \mathbf{w} = (-1,\ 11)$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3,\ v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11)

- **Linear Transformation:**

V, W : vector spaces

$T: V \rightarrow W$: A linear transformation of V into W if the following two properties are true

(1) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$

(2) $T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$

Notes:

(1) A linear transformation is said to be operation preserving

(because the same result occurs whether the operations of addition and scalar multiplication are performed before or after the linear transformation is applied)

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V | Addition in W | Scalar multiplication in V | Scalar multiplication in W

(2) A linear transformation $T: V \to V$ from a vector space into itself is called a linear operator

Example: Verifying a linear transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Proof:

$\mathbf{u} = (u_1, u_2),\ \ \mathbf{v} = (v_1, v_2)$: vector in R^2, c : any real number

(1) Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\ &= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\ &= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\ &= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$T(c\mathbf{u}) = T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)$$

$$= c(u_1 - u_2, u_1 + 2u_2)$$

$$= cT(\mathbf{u})$$

Therefore, T is a linear transformation

Example: Functions that are not linear transformations

(a) $f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$\sin(\tfrac{\pi}{2} + \tfrac{\pi}{3}) \neq \sin(\tfrac{\pi}{2}) + \sin(\tfrac{\pi}{3})$$

($f(x) = \sin x$ is not a linear transformation)

(b) $f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(1+2)^2 \neq 1^2 + 2^2$$

($f(x) = x^2$ is not a linear transformation)

(c) $f(x) = x + 1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

($f(x) = x+1$ is not a linear transformation, although it is a linear function)

In fact, $f(cx) \neq cf(x)$

Notes: Two uses of the term "linear".

(1) $f(x) = x + 1$ is called a linear function because its graph is a line

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = \mathbf{0}, \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

- Theorem: Properties of linear transformations

$$T: V \to W, \quad \mathbf{u}, \mathbf{v} \in V$$

(1) $T(\mathbf{0}) = \mathbf{0}$ ($T(c\mathbf{v}) = cT(\mathbf{v})$ for c=0)

(2) $T(-\mathbf{v}) = -T(\mathbf{v})$ ($T(c\mathbf{v}) = cT(\mathbf{v})$ for c=-1)

(3) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ ($T(\mathbf{u}+(-\mathbf{v}))=T(\mathbf{u})+T(-\mathbf{v})$ and property (2))

(4) If $\mathbf{v} = c_1v_1 + c_2v_2 + \cdots + c_nv_n$,

then $T(\mathbf{v}) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n)$

$$= c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$$

(Iteratively using $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ and $T(c\mathbf{v}) = cT(\mathbf{v})$)

Example: Linear transformations and bases

Let $T : R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$

$$T(0,1,0) = (1,5,-2)$$

$$T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$

Solution:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

(According to the fourth property on the previous slide that $T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$)

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

Example: A linear transformation defined by a matrix

The function $T: R^2 \to R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation form R^2 into R^3

Solution:

(a) $\mathbf{v} = (2, -1)$

R^2 vector R^3 vector

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore T(2, -1) = (6, 3, 0)$

(b) $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

Theorem: The linear transformation defined by a matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m

Note:

R^n vector $\quad$ R^m vector

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$T(\mathbf{v}) = A\mathbf{v} \qquad T : R^n \longrightarrow R^m$$

※ If $T(\mathbf{v})$ can represented by $A\mathbf{v}$, then T is a linear transformation

※ If the size of A is $m \times n$, then the domain of T is R^n and the codomain of T is R^m

Example: Rotations in the Plane

Show that the L.T. $T: R^2 \to R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotates every vector in R^2

counterclockwise about the origin through the angle θ

Solution:

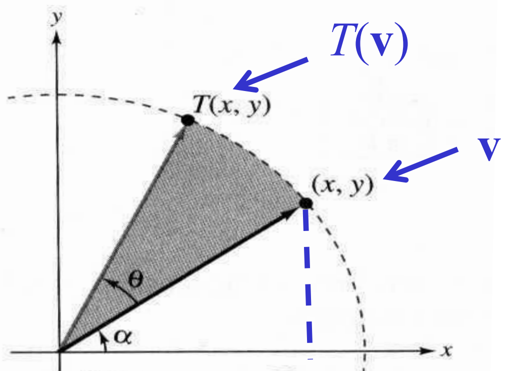
$\mathbf{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$ (Polar coordinates: for every point on the *xy*-plane, it can be represented by a set of (r, α))

r: the length of **v** ($= \sqrt{x^2 + y^2}$)

α: the angle from the positive

x-axis counterclockwise to

the vector **v**



Rotation in the Plane

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

from the addition formulas of trigonometric identities

r: remain the same, that means the length of $T(\mathbf{v})$ equals the length of **v**

$\theta+\alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector **v** counterclockwise through the angle θ

Example: The transpose function is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

$$T(A) = A^T \quad (T : M_{m\times n} \to M_{n\times m})$$

Show that T is a linear transformation

Solution:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T (the transpose function) is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$

The Kernel and Range of a Linear Transformation

- Kernel of a linear transformation T:

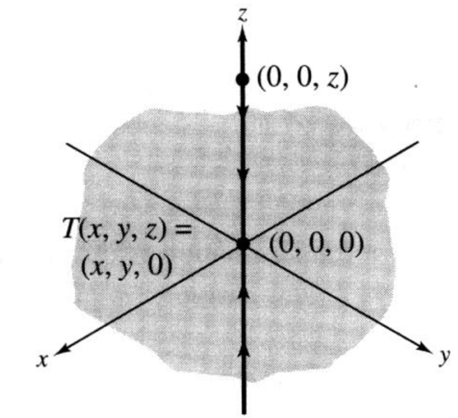
Let $T:V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the kernel of T and is denoted by ker(T)

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V\}$$

※ For example, V is R^3, W is R^3, and T is the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ Then the kernel of T is the set consisting of $(0, 0, s)$, where s is a real number, i.e.

$$\ker(T) = \{(0, 0, s) \mid s \text{ is a real number}\}$$



The kernel of T is the set of all vectors on the z-axis.

Example: Finding the kernel of a linear transformation

$$T(A) = A^T \quad (T : M_{3\times 2} \to M_{2\times 3})$$

Solution:

$$\ker(T) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: The kernel of the zero and identity transformations

(a) If $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T : V \to W$), then

$$\ker(T) = V$$

(b) If $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T : V \to V$), then

$$\ker(T) = \{\mathbf{0}\}$$

Example: Finding the kernel of a linear transformation

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \rightarrow R^2)$$

$$\ker(T) = ?$$

Solution:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0), \text{ and } (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\}$$
$$= \text{span}\{(1,-1,1)\}$$

Theorem: The kernel is a subspace of V

The kernel of a linear transformation $T: V \rightarrow W$ is a subspace of the domain V

Proof:

$\because T(\mathbf{0}) = \mathbf{0}$ (by Theorem) $\therefore \ker(T)$ is a nonempty subset of V

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T. Then

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0} \quad (\mathbf{u} \in \ker(T), \mathbf{v} \in \ker(T) \Rightarrow \mathbf{u}+\mathbf{v} \in \ker(T))$$

T is a linear transformation

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0} \quad (\mathbf{u} \in \ker(T) \Rightarrow c\mathbf{u} \in \ker(T))$$

Thus, $\ker(T)$ is a subspace of V (according to Theorem that a nonempty subset of V is a subspace of V if it is closed under vector addition and scalar multiplication)

Example: Finding a basis for the kernel

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5

Solution:

To find ker(T) means to find all $\mathbf{x}$ satisfying $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$.

Thus, we need to form the augmented matrix $[A \mid \mathbf{0}]$ first

$$[A \mid \mathbf{0}] =$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad s \qquad\qquad t$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}: \text{one basis for the kernel of } T$$

Corollary:

Let $T : R^n \rightarrow R^m$ be the linear fransformation given by $T(\mathbf{x}) = A\mathbf{x}$.

Then the kernel of T is equal to the solution space of $A\mathbf{x} = \mathbf{0}$

$$T(\mathbf{x}) = A\mathbf{x} \quad (\text{a linear transformation } T : R^n \rightarrow R^m)$$

$$\Rightarrow \ker(T) = \text{null}(A) = \left\{\mathbf{x} \mid A\mathbf{x} = 0,\ \forall \mathbf{x} \in R^n\right\} \quad (\text{subspace of } R^n)$$

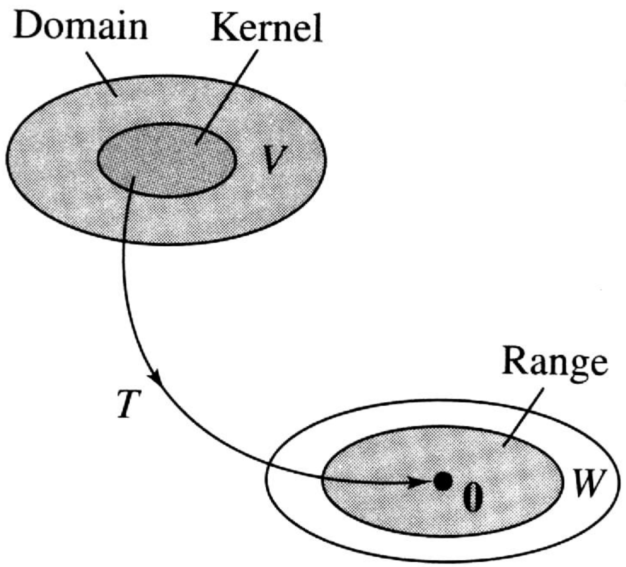
※ The kernel of T equals the null space of A and these two are both subspaces of R^n)

※ So, the kernel of T is sometimes called the null space of T

Range of a linear transformation T:

Let $T : V \rightarrow W$ be a linear transformation. Then the set of all vectors $\mathbf{w}$ in W that are images of any vectors in V is called the range of T and is denoted by range(T)

$$\text{range}(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$



※ For the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ The domain is $V=R^3$, the codomain is $W=R^3$, and the range is xy-plane (a subspace of the codomian R^3)

※ Since $T(0, 0, s) = (0, 0, 0) = \mathbf{0}$, the kernel of T is the set consisting of $(0, 0, s)$, where s is a real number

Theorem: The range of T is a subspace of W

The range of a linear transformation $T: V \to W$ is a subspace of W

Proof:

$\because T(\mathbf{0}) = \mathbf{0}$ (Theorem 6.1)

$\therefore$ range(T) is a nonempty subset of W

Since $T(\mathbf{u})$ and $T(\mathbf{v})$ are vectors in range(T), and we have

$$T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}) \in \text{range}(T)$$

(because $\mathbf{u} + \mathbf{v} \in V$; T is a linear transformation)

(Range of T is closed under vector addition because $T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{u}+\mathbf{v}) \in \text{range}(T)$)

$$cT(\mathbf{u}) = T(c\mathbf{u}) \in \text{range}(T)$$

(because $c\mathbf{u} \in V$)

(Range of T is closed under scalar multiplication because $T(\mathbf{u})$ and $T(c\mathbf{u}) \in \text{range}(T)$)

Thus, range(T) is a subspace of W (according to Theorem that a nonempty subset of W is a subspace of W if it is closed under vector addition and scalar multiplication)

Notes:

$T: V \rightarrow W$ is a linear transformation

$(1)\ \ker(T)$ is subspace of V (Theorem)

$(2)\ \text{range}(T)$ is subspace of W (Theorem)

Corollary:

Let $T: R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

The range of T is equal to the column space of A, i.e. $\text{range}(T) = \text{col}(A)$

(1) According to the definition of the range of $T(\mathbf{x}) = A\mathbf{x}$, we know that the range of T consists of all vectors **b** satisfying $A\mathbf{x}=\mathbf{b}$, which is equivalent to find all vectors **b** such that the system $A\mathbf{x}=\mathbf{b}$ is consistent

(2) $A\mathbf{x}=\mathbf{b}$ can be rewritten as

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}$$

Therefore, the system $A\mathbf{x}=\mathbf{b}$ is consistent iff we can find $(x_1, x_2, \ldots, x_n)$ such that **b** is a linear combination of the column vectors of A, i.e. $\mathbf{b} \in CS(A)$

Thus, we can conclude that the range consists of all vectors **b**, which is a linear combination of the column vectors of A or said $\mathbf{b} \in \text{col}(A)$. So, the column space of the matrix A is the same as the range of T, i.e. $\text{range}(T) = \text{col}(A)$

- Use our example to illustrate the corollary to Theorem:

※ For the orthogonal projection of any vector (x, y, z) onto the xy-plane, i.e. $T(x, y, z) = (x, y, 0)$

※ According to the above analysis, we already knew that the range of T is the xy-plane, i.e. range(T)=$\{(x, y, 0) | x$ and y are real numbers$\}$

※ T can be defined by a matrix A as follows

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ such that } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

※ The column space of A is as follows, which is just the xy-plane

$$CS(A) = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \text{ where } x_1, x_2 \in R$$

Ex 7: Finding a basis for the range of a linear transformation

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T

Sol:

Since range(T) = $col(A)$, finding a basis for the range of T is equivalent to fining a basis for the column space of A

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} \textcircled{1} & 0 & 2 & 0 & -1 \\ 0 & \textcircled{1} & -1 & 0 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix} \qquad\qquad \begin{matrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{matrix}$$

$\Rightarrow w_1,\ w_2,$ and w_4 are indepdnent, so $\{w_1,\ w_2,\ w_4\}$ can form a basis for $\text{col}(B)$

$\because$ Row operations will not affect the dependency among columns

$\therefore\ c_1,\ c_2,$ and c_4 are indepdnent, and thus $\{c_1,\ c_2,\ c_4\}$ is a basis for $\text{col}(A)$

That is, $\{(1,\ 2,\ -1,\ 0),\ (2,\ 1,\ 0,\ 0),\ (1,\ 1,\ 0,\ 2)\}$ is a basis for the range of T

- Rank of a linear transformation $T: V \rightarrow W$:

$$\text{rank}(T) = \text{the dimension of the range of } T = \dim(\text{range}(T))$$

According to the corollary to Thm., range(T) = col(A), so dim(range(T)) = dim(col(A))

- Nullity of a linear transformation $T: V \rightarrow W$:

$$\text{nullity}(T) = \text{the dimension of the kernel of } T = \dim(\ker(T))$$

According to the corollary to Thm., ker(T) = $null$(A), so dim(ker(T)) = dim($null$(A))

- Note:

If $T: R^n \rightarrow R^m$ is a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$\text{rank}(T) = \dim(\text{range}(T)) = \dim(\text{col}(A)) = \text{rank}(A)$$

$$\text{nullity}(T) = \dim(\ker(T)) = \dim(\text{null}(A)) = \text{nullity}(A)$$

※ The dimension of the row (or column) space of a matrix A is called the rank of A

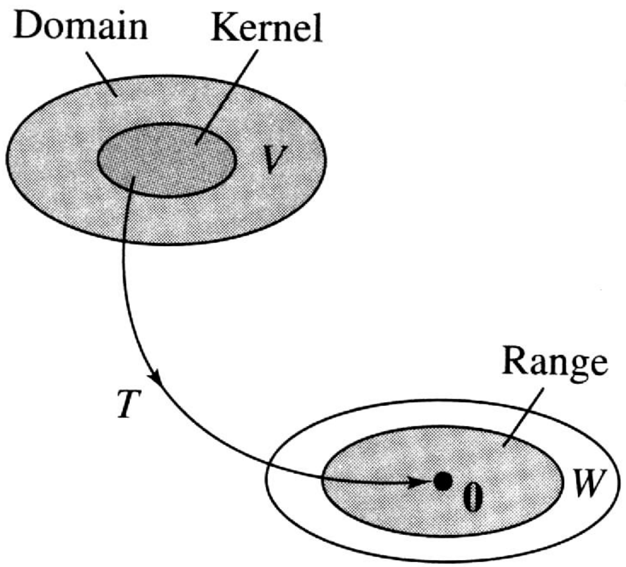
※ The dimension of the nullspace of A ($\text{null}(A) = \{\mathbf{x} \mid A\mathbf{x} = 0\}$) is called the nullity of A

Theorem: Sum of rank and nullity

Let $T: V \rightarrow W$ be a linear transformation from an n-dimensional vector space V (i.e. the dim(domain of T) is n) into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$(\text{i.e. } \dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T))$$



※ You can image that the dim(domain of T) should equals the dim(range of T) originally

※ But some dimensions of the domain of T is absorbed by the zero vector in W

※ So the dim(range of T) is smaller than the dim(domain of T) by the number of how many dimensions of the domain of T are absorbed by the zero vector, which is exactly the dim(kernel of T)

Proof:

Let T be represented by an $m \times n$ matrix A, and assume $\text{rank}(A) = r$

$$(1)\ \text{rank}(T) = \dim(\text{range of } T) = \dim(\text{column space of } A) = \text{rank}(A) = r$$

$$(2)\ \text{nullity}(T) = \dim(\text{kernel of } T) = \dim(\text{null space of } A) = n - r$$

$$\Rightarrow \text{rank}(T) + \text{nullity}(T) = r + (n - r) = n$$

according to Thm. where $\text{rank}(A) + \text{nullity}(A) = n$

※ Here we only consider that T is represented by an $m \times n$ matrix A. In the next section, we will prove that any linear transformation from an n-dimensional space to an m-dimensional space can be represented by $m \times n$ matrix

Example: Finding the rank and nullity of a linear transformation

Find the rank and nullity of the linear transformation $T: R^3 \rightarrow R^3$ define by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

※ The rank is determined by the number of leading 1's, and the nullity by the number of free variables (columns without leading 1's)

Example: Finding the rank and nullity of a linear transformation

Let $T: R^5 \to R^7$ be a linear transformation

(a) Find the dimension of the kernel of T if the dimension of the range of T is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Solution:

(a) $\dim(\text{domain of } T) = n = 5$

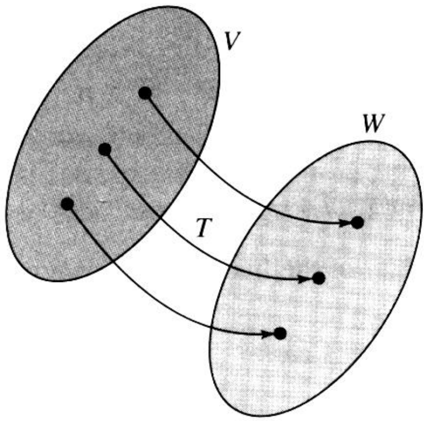
$\dim(\text{kernel of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

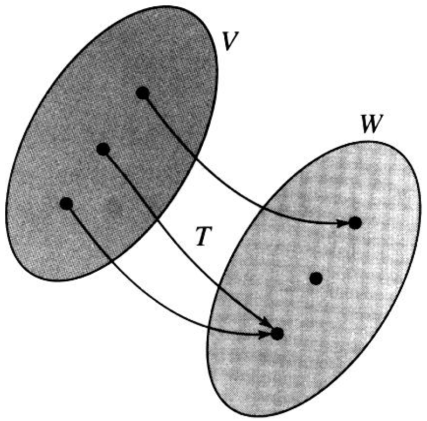
(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

- One-to-one:

A function $T: V \rightarrow W$ is called one-to-one if the preimage of every $\mathbf{w}$ in the range consists of a single vector. This is equivalent to saying that T is one-to-one iff for all $\mathbf{u}$ and $\mathbf{v}$ in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$



one-to-one



not one-to-one

Theorem: One-to-one linear transformation

Let $T: V \to W$ be a linear transformation. Then

T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

Proof:

$(\Rightarrow)$ Suppose T is one-to-one

Then $T(\mathbf{v}) = \mathbf{0}$ can have only one solution : $\mathbf{v} = \mathbf{0}$

Due to the fact that $T(\mathbf{0}) = \mathbf{0}$ in Thm.

i.e. $\ker(T) = \{\mathbf{0}\}$

$(\Leftarrow)$ Suppose $\ker(T)=\{\mathbf{0}\}$ and $T(\mathbf{u})=T(\mathbf{v})$

$T(\mathbf{u}-\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$

T is a linear transformation, see Property 3 in Thm. 6.1

$\therefore \mathbf{u}-\mathbf{v} \in \ker(T) \Rightarrow \mathbf{u}-\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{v}$

$\Rightarrow T$ is one-to-one (because $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$)

Example: One-to-one and not one-to-one linear transformation

(a) The linear transformation $T: M_{m\times n} \rightarrow M_{n\times m}$ given by $T(A) = A^T$ is one-to-one

because its kernel consists of only the $m\times n$ zero matrix

(b) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one

because its kernel is all of R^3

- Onto:

A function $T: V \to W$ is said to be onto if every element in W has a preimage in V

(T is onto W when W is equal to the range of T)

Theorem: Onto linear transformations

Let T: $V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the rank of T is equal to the dimension of W

$$\text{rank}(T) = \dim(\text{range of } T) = \dim(W)$$

↓ The definition of the rank of a linear transformation

↓ The definition of onto linear transformations

Theorem: One-to-one and onto linear transformations

Let $T: V \to W$ be a linear transformation with vector space V and W both of dimension n. Then T is one-to-one if and only if it is onto

Proof:

$(\Rightarrow)$ If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$ and $\dim(\ker(T)) = 0$

$$\dim(\text{range}(T)) \overset{\text{Thm.}}{=} n - \dim(\ker(T)) = n = \dim(W)$$

Consequently, T is onto

According to the definition of dimension that if a vector space V consists of the zero vector alone, the dimension of V is defined as zero

$(\Leftarrow)$ If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$$\dim(\ker(T)) \overset{\text{Thm. 6.5}}{=} n - \dim(\text{range of } T) = n - n = 0 \Rightarrow \ker(T) = \{\mathbf{0}\}$$

Therefore, T is one-to-one

Example:

The linear transformation $T: R^n \rightarrow R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:

$T:R^n \rightarrow R^m$	dim(domain of T) (1)	rank(T) (2)	nullity(T)	1-1	onto
	$= \dim(R^n) = n$	$= \dim(\text{range of } T) = \#$ of leading 1's	$= (1) - (2) = \dim(\ker(T))$	If nullity(T) $= \dim(\ker(T)) = 0$	If rank(T) $= \dim(R^m) = m$
(a) $T:R^3 \rightarrow R^3$	3	3	0	Yes	Yes
(b) $T:R^2 \rightarrow R^3$	2	2	0	Yes	No
(c) $T:R^3 \rightarrow R^2$	3	2	1	No	Yes
(d) $T:R^3 \rightarrow R^3$	3	2	1	No	No

Definition: Isomorphism

A linear transformation $T: V \to W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other

Theorem: Isomorphic spaces and dimension

Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension

Proof:

$(\Rightarrow)$ Assume that V is isomorphic to W, where V has dimension n

$\Rightarrow$ There exists a L.T. $T: V \to W$ that is one to one and onto

$\because T$ is one-to-one

$\Rightarrow \dim(\ker(T)) = 0$

$\dim(V) = n$

$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(\ker(T)) = n - 0 = n$

$\because T$ is onto

$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$

Thus $\dim(V) = \dim(W) = n$

$(\Leftarrow)$ Assume that V and W both have dimension n

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be a basis of V and

$B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ be a basis of W

Then an arbitrary vector in V can be represented as

$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$

and you can define a L.T. $T: V \to W$ as follows

$\mathbf{w} = T(\mathbf{v}) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$ (by defining $T(\mathbf{v}_i) = \mathbf{w}_i$)

Since B' is a basis for V, $\{\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n\}$ is linearly independent, and the only solution for $\mathbf{w}=\mathbf{0}$ is $c_1=c_2=\ldots=c_n=0$

So with $\mathbf{w}=\mathbf{0}$, the corresponding $\mathbf{v}$ is $\mathbf{0}$, i.e., $\ker(T) = \{\mathbf{0}\}$

$\Rightarrow T$ is one - to - one

By Theorem 6.5, we can derive that dim(range of T) = dim(domain of T) – dim(ker(T)) = $n - 0 = n = \dim(W)$

$\Rightarrow T$ is onto

Since this linear transformation is both one-to-one and onto, then V and W are isomorphic

Note

The above Theorem tells us that every vector space with dimension n is isomorphic to R^n

Ex 12: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other

(a) $R^4 = 4\text{-space}$

(b) $M_{4\times 1} =$ space of all 4×1 matrices

(c) $M_{2\times 2} =$ space of all 2×2 matrices

(d) $P_3(x) =$ space of all polynomials of degree 3 or less

(e) $V = \{(x_1,\ x_2,\ x_3,\ x_4,\ 0), x_i$ are real numbers$\}$ (a subspace of R^5)

Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \to R^3$:

$$(1)\ T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write
  - It is simpler to read
  - It is more easily adapted for computer use

Theorem: Standard matrix for a linear transformation

Let $T: R^n \rightarrow R^m$ be a linear trtansformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathrm{n}\}$ is a standard basis for R^n. Then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$,

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n, A is called the standard matrix for T

Proof:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

$$T \text{ is a linear transformation} \Rightarrow T(\mathbf{v}) = T(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n)$$

$$= T(v_1\mathbf{e}_1) + T(v_2\mathbf{e}_2) + \cdots + T(v_n\mathbf{e}_n)$$

$$= v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + \cdots + v_n T(\mathbf{e}_n)$$

$$\text{If } A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ \cdots\ T(\mathbf{e}_n) \right], \text{ then}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(\mathbf{e}_1) + v_2 T(\mathbf{e}_2) + \cdots + v_n T(\mathbf{e}_n)$$

$$\text{Therefore, } T(\mathbf{v}) = A\mathbf{v} \text{ for each } \mathbf{v} \text{ in } R^n$$

Note

Theorem tells us that once we know the image of every vector in the standard basis (that is $T(\mathbf{e}_i)$), you can use the properties of linear transformations to determine $T(\mathbf{v})$ for any $\mathbf{v}$ in V

Example: Finding the standard matrix of a linear transformation

Find the standard matrix for the L.T. $T: R^3 \to R^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Solution:

Vector Notation	Matrix Notation
$T(\mathbf{e}_1) = T(1,\ 0,\ 0) = (1,\ 2)$	$T(\mathbf{e}_1) = T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
$T(\mathbf{e}_2) = T(0,\ 1,\ 0) = (-2,\ 1)$	$T(\mathbf{e}_2) = T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$
$T(\mathbf{e}_3) = T(0,\ 0,\ 1) = (0,\ 0)$	$T(\mathbf{e}_3) = T(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3) \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

$$\text{i.e., } T(x, y, z) = (x-2y, 2x+y)$$

- Note: a more direct way to construct the standard matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x-2y+0z \\ \leftarrow & 2x+1y+0z \end{matrix}$$

※ The first (second) row actually represents the linear transformation function to generate the first (second) component of the target vector

Example: Finding the standard matrix of a linear transformation

The linear transformation $T : R^2 \rightarrow R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T

Solution:

$$T(x, y) = (x, 0)$$

$$A = \left[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\right] = \left[T(1,\ 0)\ T(0,\ 1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Notes:

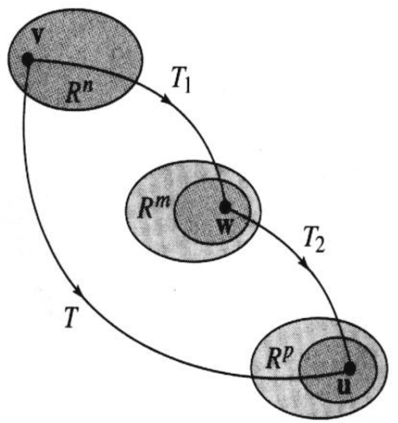
(1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix

(2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n

- Composition of $T_1:R^n \to R^m$ with $T_2:R^m \to R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

This composition is denoted by $T = T_2 \circ T_1$



Composition of Transformations

Theorem: Composition of linear transformations

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2 ,then

(1) The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is still a linear transformation

(2) The standard matrix A for T is given by the matrix product $A = A_2 A_1$

Proof:

(1) (T is a linear transformation)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar. Then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u}))+T_2(T_1(\mathbf{v})) = T(\mathbf{u})+T(\mathbf{v})$$
$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2) (A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

Note:

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

Example: The standard matrix of a composition

Let T_1 and T_2 be linear transformations from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, \text{z}, y)$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$

Solution:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- The matrix of *T* relative to the bases *B* and *B'*:

$T: V \rightarrow W$ (a linear transformation)

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ (a nonstandard basis for V)

The coordinate matrix of any $\mathbf{v}$ relative to B is denoted by $[\mathbf{v}]_B$

$$\left(\text{if } \mathbf{v} \text{ can be represeted as } c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \text{ then } [\mathbf{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right)$$

A matrix *A* can represent *T* if the result of *A* multiplied by a coordinate matrix of **v** relative to *B* is a coordinate matrix of **v** relative to *B'*, where *B'* is a basis for *W*. That is,

$$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B,$$

where *A* is called the matrix of *T* relative to the bases *B* and *B'*

- Transformation matrix for nonstandard bases (the generalization of Theorem, in which standard bases are considered) :

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$

If $T: V \rightarrow W$ is a linear transformation s.t.

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(v_i)]_{B'}$

$$A = \big[[T(\mathbf{v}_1)]_{B'} \ [T(\mathbf{v}_2)]_{B'} \ \cdots \ [T(\mathbf{v}_n)]_{B'}\big] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\big[T(\mathbf{v})\big]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V

※The above result state that the coordinate of $T(\mathbf{v})$ relative to the basis B' equals the multiplication of A defined above and the coordinate of $\mathbf{v}$ relative to the basis B.

※ Comparing to the result in Thm. ($T(\mathbf{v}) = A\mathbf{v}$), it can infer that the linear transformation and the basis change can be achieved in one step through multiplying the matrix A defined above

Example: Finding a matrix relative to nonstandard bases

Let $T : R^2 \to R^2$ be a linear transformation defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis $B = \{(1,\ 2),\ (-1,\ 1)\}$ and $B' = \{(1,\ 0),\ (0,\ 1)\}$

Solution:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$[T(1, 2)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad [T(-1, 1)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[[T(1, 2)]_{B'} \quad [T(1, 2)]_{B'} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

Example:

For the L.T. $T: R^2 \rightarrow R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2,\ 1)$

Solution:

$$\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1) \qquad B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3) \qquad B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$$

Notes:

(1) In the special case where $V = W$ (i.e., $T : V \to V$) and $B = B'$,

the matrix A is called the matrix of T relative to the basis B

(2) If $T : V \to V$ is the identity transformation

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A = \left[[T(\mathbf{v}_1)]_B \ [T(\mathbf{v}_2)]_B \ \cdots \ [T(\mathbf{v}_n)]_B \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

Transition Matrices and Linear Transformations

$$T : V \rightarrow V \qquad \text{(a linear transformation)}$$

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \qquad \text{(a basis of } V\text{)}$$

$$B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\} \qquad \text{(a basis of } V\text{)}$$

$$A = \Big[[T(\mathbf{v}_1)]_B \ \ [T(\mathbf{v}_2)]_B \ \ \cdots \ \ [T(\mathbf{v}_n)]_B \Big] \ \text{(matrix of } T \text{ relative to } B\text{)}$$

$$A' = \Big[[T(\mathbf{w}_1)]_{B'} \ \ [T(\mathbf{w}_2)]_{B'} \ \ \cdots \ \ [T(\mathbf{w}_n)]_{B'} \Big] \ \text{(matrix of } T \text{ relative to } B'\text{)}$$

$$\therefore [T(\mathbf{v})]_B = A[\mathbf{v}]_B, \text{ and } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$

$${}_B P_{B'} = \Big[[\mathbf{w}_1]_B \ \ [\mathbf{w}_2]_B \ \ \cdots \ \ [\mathbf{w}_n]_B \Big] \ \text{(transition matrix from } B' \text{ to } B\text{)}$$

$${}_B P_{B'}{}^{-1} = {}_{B'} P_B = \Big[[\mathbf{v}_1]_{B'} \ \ [\mathbf{v}_2]_{B'} \ \ \cdots \ \ [\mathbf{v}_n]_{B'} \Big] \ \text{(transition matrix from } B \text{ to } B'\text{)}$$

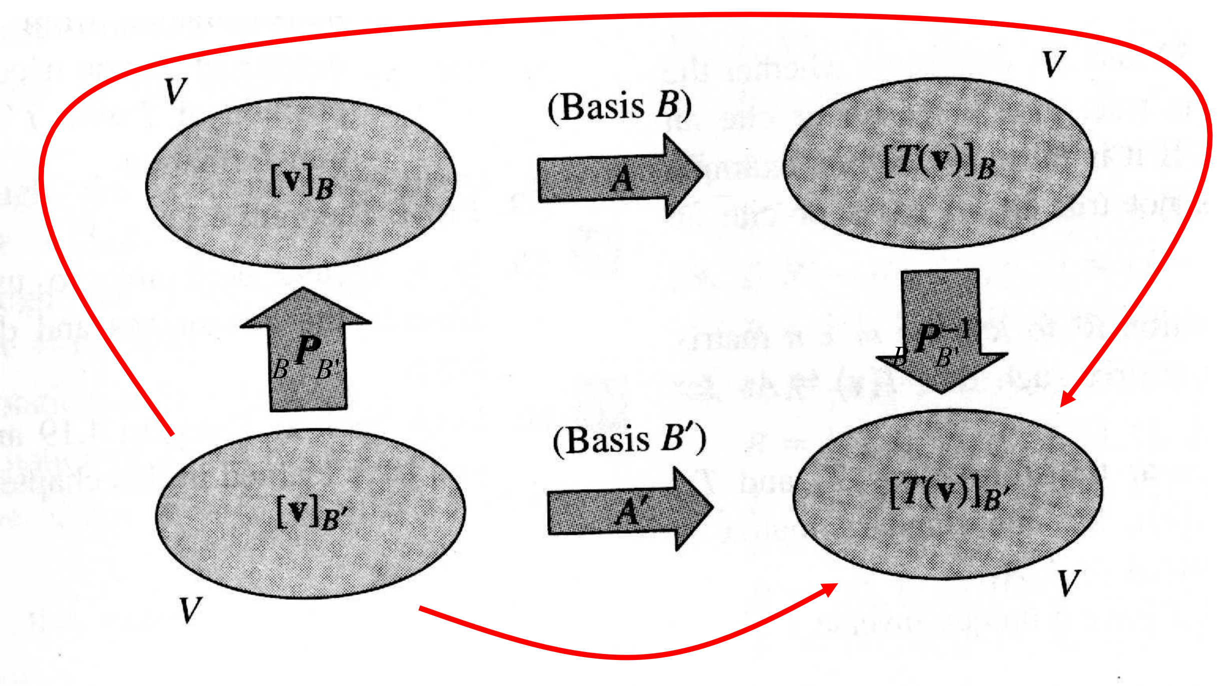
$$\therefore [\mathbf{v}]_B = {}_B P_{B'} [\mathbf{v}]_{B'}, \text{ and } [\mathbf{v}]_{B'} = {}_{B'} P_B [\mathbf{v}]_B$$

- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

(1) (direct): $A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$

(2) (indirect): ${}_{B'}P_B A \, {}_B P_{B'} [\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'} \Rightarrow A' = {}_{B'}P_B A \, {}_B P_{B'}$

indirect



direct

Example: Finding a matrix for a linear transformation

Find the matrix A' for $T: R^2 \to R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

reletive to the basis $B' = \{(1, 0), (1, 1)\}$

Solution:

$$(1)\ A' = \left[[T(1,\ 0)]_{B'} \quad [T(1,\ 1)]_{B'} \right]$$

$$T(1,0) = (2,-1) = 3(1,0) - 1(1,1) \ \Rightarrow \ [T(1,0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1,1) = (0,2) = -2(1,0) + 2(1,1) \ \Rightarrow \ [T(1,1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = [[T(1,0)]_{B'} \quad [T(1,1)]_{B'}] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(2) standard matrix for T (matrix of T relative to $B = \{(1,\ 0),\ (0,\ 1)\}$)

$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$_{B}P_{B'} = \begin{bmatrix} [(1,\ 0)]_B & [(1,\ 1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

※ Solve $a(1, 0) + b(0, 1) = (1, 0) \Rightarrow$ $(a, b) = (1, 0)$

※ Solve $c(1, 0) + d(0, 1) = (1, 1) \Rightarrow$ $(c, d) = (1, 1)$

transition matrix from B to B'

$$_{B'}P_{B} = \begin{bmatrix} [(1,\ 0)]_{B'} & [(0,\ 1)]_{B'} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

※ Solve $a(1, 0) + b(1, 1) = (1, 0) \Rightarrow$ $(a, b) = (1, 0)$

※ Solve $c(1, 0) + d(1, 1) = (0, 1) \Rightarrow$ $(c, d) = (-1, 1)$

matrix of T relative B'

$$A' = {_{B'}P_B}\, A\, {_BP_{B'}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

Example: Finding a matrix for a linear transformation

Let $B=\{(-3,\ 2),\ (4,\ -2)\}$ and $B'=\{(-1,\ 2),\ (2,\ -2)\}$ be basis for R^2, and let $A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}$ be the matrix for $T:R^2\to R^2$ relative to B.

Find the matrix of T relative to B'

Solution:

Because the specific function is unknown, it is difficult to apply the direct method to derive A', so we resort to the indirect method where $A' = {}_{B'}P_B\, A\, {}_BP_{B'}$

transition matrix from B' to B: ${}_BP_{B'}=\begin{bmatrix}[(-1,\ 2)]_B & [(2,\ -2)]_B\end{bmatrix}=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$

transition matrix from B to B': ${}_{B'}P_B=\begin{bmatrix}[(-3,\ 2)]_{B'} & [(4,\ -2)]_{B'}\end{bmatrix}=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}$

matrix of T relative to B':

$$A'={}_{B'}P_B\,A\,{}_BP_{B'}=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$$

Example: (Finding a matrix for a linear transformation)

For the linear transformation $T: R^2 \rightarrow R^2$ given in last Example, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$, for the vector **v** whose coordinate matrix is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Solution:

$$[\mathbf{v}]_B = {}_BP_{B'}[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = {}_{B'}P_B[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

or $$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$