

Section 8.5

Integration of Rational Functions by Partial Fractions

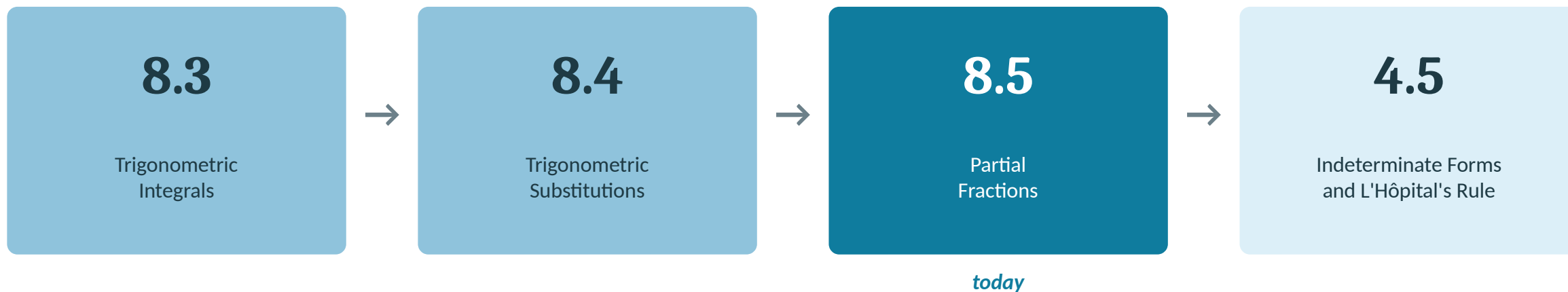
Breaking one hard fraction into several easy ones

Math 111 • Integral Calculus

Thomas' Calculus: Early Transcendentals, 15th ed. (Global Edition)

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Where This Lecture Fits



The whole idea, in one line

$$(5x - 3) / (x^2 - 2x - 3) = 2 / (x + 1) + 3 / (x - 3)$$

The left-hand side has no obvious antiderivative. The right-hand side is two logarithms. Today's work is turning the first into the second — and that work is algebra, not calculus.



Before You Start

Three things must be true, or the method will not work.

1

The fraction must be proper

degree of the numerator $<$ degree of the denominator. If not, divide first — Example 3.

2

You must be able to factor the denominator

into real linear factors and irreducible quadratic factors.

3

You must solve for the coefficients

either by equating like powers of x , or by substituting convenient values of x .

A quadratic is irreducible when it has no real roots — it cannot be split into two real linear factors.

∫ Which Factor Gets Which Form

Two rules decide the whole decomposition.

A linear factor $(x - r)^m$

gives m fractions, one for each power:

$$A_1 / (x - r) + A_2 / (x - r)^2 + \dots + A_m / (x - r)^m$$

An irreducible quadratic $(x^2 + px + q)^n$

gives n fractions, each with a linear numerator:

$$(B_1x + C_1) / (x^2 + px + q) + \dots + (B_nx + C_n) / (x^2 + px + q)^n$$

The count is fixed. A factor raised to the power m contributes exactly m unknowns, and a quadratic to the power n contributes $2n$. Add them up before you start — that is how many equations you will need.

Example: $(x - 2) / [(x + 1)^2 (x^2 + 4)]$ needs $2 + 2 = 4$ unknowns

Example 1 — distinct linear factors

Evaluate $\int (x^2 + 4x + 1) / [(x - 1)(x + 1)(x + 3)] dx$

$$(x^2 + 4x + 1) / [(x - 1)(x + 1)(x + 3)] = A/(x - 1) + B/(x + 1) + C/(x + 3)$$

Clear fractions: $x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$

Now substitute the root of each factor — two terms vanish each time:

$$\begin{aligned} x &= 1 \\ 6 &= 8A \\ A &= 3/4 \end{aligned}$$

$$\begin{aligned} x &= -1 \\ -2 &= -4B \\ B &= 1/2 \end{aligned}$$

$$\begin{aligned} x &= -3 \\ -2 &= 8C \\ C &= -1/4 \end{aligned}$$

$$= (3/4) \ln |x - 1| + (1/2) \ln |x + 1| - (1/4) \ln |x + 3| + K$$



Example 2 — a repeated linear factor

Evaluate $\int (6x + 7) / (x + 2)^2 dx$

$(x + 2)$ is squared, so it contributes two fractions — one for each power.

$$(6x + 7) / (x + 2)^2 = A/(x + 2) + B/(x + 2)^2$$

set up the form

$$6x + 7 = A(x + 2) + B = Ax + (2A + B)$$

clear fractions

$$A = 6 \text{ and } 2A + B = 7 \Rightarrow B = -5$$

equate like powers of x

$$\int [6(x + 2)^{-1} - 5(x + 2)^{-2}] dx$$

now both terms are standard

$$= 6 \ln | x + 2 | + 5 / (x + 2) + C$$

Example 3 — an improper fraction

Evaluate $\int (2x^3 - 4x^2 - x - 3) / (x^2 - 2x - 3) dx$

The numerator has degree 3 and the denominator degree 2, so the fraction is improper. Divide first.

$$(2x^3 - 4x^2 - x - 3) / (x^2 - 2x - 3) = 2x + (5x - 3) / (x^2 - 2x - 3)$$

long division

$$(5x - 3) / [(x + 1)(x - 3)] = 2/(x + 1) + 3/(x - 3)$$

the opening example

$$\int [2x + 2/(x + 1) + 3/(x - 3)] dx$$

every term is now standard

$$= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C$$



Try these first

Write down the correct form — you need not solve for the coefficients yet.

(a) $(-2x + 4) / [(x^2 + 1)(x - 1)^2]$

how many unknowns?

(b) $1 / [x(x^2 + 1)^2]$

how many unknowns?

Count the factors and their powers first. Setting up the right form is most of the work.

Example 4 — a quadratic and a repeated linear factor

Evaluate $\int (-2x + 4) / [(x^2 + 1)(x - 1)^2] dx$

$$(-2x + 4) / [(x^2 + 1)(x - 1)^2] = (Ax + B)/(x^2 + 1) + C/(x - 1) + D/(x - 1)^2$$

Clearing fractions and collecting powers of x gives four equations:

$$x^3: \quad 0 = A + C$$

$$x^2: \quad 0 = -2A + B + C + D$$

$$x^1: \quad -2 = A - 2B + C$$

$$x^0: \quad 4 = B - C + D$$

$$\text{Solving: } A = 2, \quad B = 1, \quad C = -2, \quad D = 1$$

$$\int [(2x + 1)/(x^2 + 1) - 2/(x - 1) + 1/(x - 1)^2] dx$$

$$= \ln(x^2 + 1) + \arctan x - 2 \ln |x - 1| - 1/(x - 1) + K$$

Example 5 — a repeated irreducible quadratic

Evaluate $\int dx / [x(x^2 + 1)^2]$

$$1 / [x(x^2+1)^2] = A/x + (Bx + C)/(x^2+1) + (Dx + E)/(x^2+1)^2$$

Multiply through and collect powers of x

five equations in five unknowns

$$A = 1, \quad B = -1, \quad C = 0, \quad D = -1, \quad E = 0$$

the solution

$$\int [1/x - x/(x^2+1) - x/(x^2+1)^2] dx$$

both quadratic terms take $u = x^2 + 1$

$$= \ln |x| - \frac{1}{2} \ln (x^2+1) + 1 / [2(x^2+1)] + K$$

Example 6 — finding the coefficients by differentiating

Find A, B and C in $(x - 1) / (x + 1)^3 = A/(x+1) + B/(x+1)^2 + C/(x+1)^3$

Clear fractions: $x - 1 = A(x + 1)^2 + B(x + 1) + C$

Put $x = -1$

$$-2 = C$$

two terms vanish

Differentiate once: $1 = 2A(x + 1) + B$, then put $x = -1$

$$1 = B$$

one more vanishes

Differentiate again: $0 = 2A$

$$A = 0$$

and the last falls out

$$(x - 1) / (x + 1)^3 = 1 / (x + 1)^2 - 2 / (x + 1)^3$$

Summary

- 1 Check the degrees first — if the fraction is improper, divide before anything else.
- 2 Factor the denominator — into real linear and irreducible quadratic factors.
- 3 Write the right form — $(x - r)^m$ gives m fractions; a quadratic to the power n gives n fractions with linear numerators.
- 4 Solve for the coefficients — substitute the roots, equate like powers, or differentiate — whichever is quickest.
- 5 Then integrate term by term — every piece is a logarithm, a power, or an arctangent.

Practice

Exercises 8.5

1–8 just decompose

9–16 distinct linear

17–20 repeated linear

21–32 quadratic factors

Write the form before you solve for anything.

Next lecture: §4.5 — Indeterminate Forms and L'Hôpital's Rule