The Radon-Nikodym Theorem

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The Riez Theorem

Let H be real a Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $\|\ \|$.

Theorem

(Projection Theorem)
If K is a closed subspace of H, x a vector and $d = \inf\{\|y - x\|;\ y \in K\}$, then there exists a unique vector $z \in K$ such that $\|z - x\| = d$. Moreover $z - x$ is orthogonal to K. (z is called the orthogonal projection of x on K and denoted by $p_K(x)$)

Proof

Let $(x_n)_n$ be a sequence of K such that $\lim_{n\to+\infty} ||x - x_n|| = d$. It follows from the parallelogram law ($||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$) applied to the vectors $x - x_n$ and $x - x_m$ that

$$||x_n - x_m||^2 = 2(||x_n - x||^2 + ||x_m - x||^2) - ||x_n + x_m - 2x||^2$$

$\lim_{n\to+\infty} ||x_n - x||^2 = d^2$ and $||x_n + x_m - 2x||^2 = 4||\frac{x_n+x_m}{2} - x||^2 \geq 4d^2$ because $\frac{x_n+x_m}{2} \in K$. Then $(x_n)_n$ is a Cauchy sequence in K which is complete, then it converges in K. If $\lim_{n\to+\infty} x_n = z \in K$ then by continuity $||x - y|| = d$. If $z' \in K$ is such that $||x - z'|| = d$, then from the parallelogram law

$$||z - z'||^2 = 2(||z - x||^2 + ||z' - x||^2) - 4||\frac{z + z'}{2} - x||^2,$$

which proves that $||z - z'|| \leq 0$ and $z = z'$.
Let $t \in \mathbb{R}$ and $y \in K^*$, $p_K(x) - ty \in K$, therefore

$$\|x - p_K(x)\|^2 \leq \|x - p_K(x) - ty\|^2.$$

This inequality simplifies to

$$t^2\|y\|^2 - 2t\langle y, x - p_K(x)\rangle \geq 0.$$

Taking in particular $t = \dfrac{\langle x - p_K(x), y\rangle}{\|y\|^2}$, it obtains that $0 \leq -\langle x - p_K(x), y\rangle^2\|y\|^2$, hence that $x - p_K(x) \perp K$. $\square$

Corollary

(Orthogonal Decomposition)
Let K be a closed subspace of H. Then every vector x can be written in a unique way as a sum

$$x = T(x) + P(x),$$

where $T(x) \in K$ and $P(x) \perp K$.

Theorem

(Riesz Representation)
If L is a continuous linear functional on H, then there exists a unique vector $z \in H$ such that

$$L(x) = \langle x, z \rangle, \quad \forall\, x \in H.$$

Proof

If $L(x) = 0$ for every $x \in H$, then take $z = 0$.
Assume thus L is not identically 0. Let $K = \{x \in H;\ L(x) = 0\}$. Then K is a closed subspace of H. Since $L \neq 0$, then $K \neq H$. Thus, by Corollary (6), there exits a non-zero vector $y \in H$ such that $y \in K^{\perp}$. It may be assumed that $\|y\| = 1$. Put $u = L(x)y - L(y)x$. Then $u \in K$ because $L(u) = L(x)L(y) - L(y)L(x) = 0$. Therefore $\langle u, y\rangle = 0$. But $\langle u, y\rangle = \langle L(x)y - L(y)x, y\rangle = L(x) - L(y)\langle x, y\rangle$ so that $L(x) = L(y)\langle x, y\rangle$. Hence, $L(x) = \langle x, z\rangle$ with $z = L(y)y$. To see that z is unique, note that if $\langle x, z\rangle = \langle x, z'\rangle$ for all $x \in H$, then $u = z - z'$ is such that $\langle x, u\rangle = 0$ for all $x \in H$, hence $u = 0$. $\square$

Lebesgue Decomposition Theorem and Radon-Nikodym Theorem

Definition

1. Let $(X, \mathscr{A})$ be a measurable space and let μ and ν be two measures on $(X, \mathscr{A})$. We say that ν is absolutely continuous with respect to μ, in symbols $\nu << \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$.
2. A measure on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$ is called absolutely continuous if it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$.

Remark

If f is a non-negative $\mu-$integrable function on X, then $\nu(E) = \int_E f(x)d\mu(x)$ defines a measure on X which is absolutely continuous with respect to μ.

Definition

Let $(X, \mathscr{A})$ be a measurable space and let μ and ν be two measures on $(X, \mathscr{A})$.

• The measure ν is said to be concentrated on E if $\nu(A) = \nu(E \cap A)$ for all $A \in \mathscr{A}$.

• μ and ν are said to be mutually singular, in symbols $\mu \perp \nu$, if there exist disjoint measurable subsets E and F such that $X = E \cup F$ and $\mu(E) = 0$, $\nu(F) = 0$.

Theorem

(Lebesgue Decomposition Theorem and Radon-Nikodym Theorem)
Let $(X, \mathscr{A}, \mu)$ be a measurable space and μ $\sigma-$finite. If ν is a $\sigma-$finite measure on $(X, \mathscr{A})$, there exist unique measures ν_a and ν_s such that $\nu = \nu_a + \nu_s$, $\nu_a << \mu$ and $\nu_s \perp \mu$.
Moreover there exits a unique $g \in L^1(X, \mathscr{A}, \mu)$ such that

$$\nu_a(A) = \int_A g(x) d\mu(x), \quad \forall\ A \in \mathscr{A}.$$

Proof

Uniqueness

Let (ν_a, ν_s) and (ν'_a, ν'_s) be two solutions, then

$$\forall A \in \mathscr{A}, \quad \nu_a(A) - \nu'_a(A) = \nu'_s(A) - \nu_s(A).$$

Since ν_s and ν'_s are singular with respect to μ, there exist two μ null sets E and E' such that $\nu_s(E^c) = 0$ and $\nu'_s(E'^c) = 0$ and

$$\forall A \in \mathscr{A}, \quad \nu_s(A) - \nu'_s(A) = \nu_s(A \cap (E \cup E')) - \nu'_s(A \cap (E \cup E')) = \nu'_a(A \cap (E \cup$$

since $\nu_a << \mu$ and $\nu'_a << \mu$.

To obtain the uniqueness of g, let g' an other solution, then

$$\int_{\{t;\ g'(t) > g(t)\}} g'(x) d\mu(x) = \nu_a(\{t;\ g'(t) > g(t)\}) = \int_{\{t;\ g'(t) > g(t)\}} g(x)$$

Then

$$\int (g'(x) - g(x)) d\mu(x) = 0$$

Existence In the first case we assume that the measures μ and ν are finite. Let H be the Hilbert space $H = L^2(X, \mathscr{A}, \mu + \nu)$. Since the measures μ and ν are finite, so is $\mu + \nu$. Moreover, if $f \in H$, by Hölder inequality

$$\left|\int_X f(x)d\nu(x)\right| \leq \int_X |f(x)|d(\mu+\nu)(x) \leq \left(\int_X f^2(x)d\nu(x)\right)^{\frac{1}{2}} (\mu+\nu)(X))$$

then $f \in L^2(X, \nu)$ and

$$f \in H \longmapsto \int_X f(x)d\nu(x)$$

is a continuous linear functional on H, because $\nu \leq \mu + \nu$. By Theorem (6), there exists $g \in H$ such that

$$\int_X f(x)d\nu(x) = \int_X f(x)g(x)d(\mu + \nu)(x),$$

for all $f \in H$. In particular for all $A \in \mathscr{A}$ such that $(\mu + \nu)(A) \neq 0$

For $\varepsilon > 0$ let $F = \{x;\ g(x) > \varepsilon\}$, $(\mu + \nu)(F) \geq \int_F g(x)d(\mu + \nu)(x) \geq (1 + \varepsilon)(mu + \nu)(F) \Rightarrow (mu + \nu)(F) = 0$.
Let $G = \{x;\ g(x) < -\varepsilon\}$, $0 \geq \int_G g(x)d(\mu + \nu)(x) \leq -\varepsilon(mu + \nu)(G) \Rightarrow (mu + \nu)(G) = 0$.
We can assume that $0 \leq g \leq 1$ on X. Then

$$\int_X (1-g)(x)f(x)d\nu(x) = \int_X f(x)g(x)d\mu(x), \quad \forall f \in L^2(X, \mathscr{A}, \mu+\nu). \tag{1}$$

We set $A = \{x \in X;\ 0 \leq g(x) < 1\}$ and $B = \{x \in X;\ g(x) = 1\}$, $\nu_a(E) = \nu(E \cap A)$ and $\nu_s(E) = \nu(E \cap B)$ for all $E \in \mathscr{A}$. We apply the equation (1) to $f = \chi_B$, we find $\mu(B) = 0$ which proves that $\nu_s \perp \mu$.

We apply the equation (1) to $f = (1 + g + \ldots + g^n)\chi_E$, we find

$$\int_E (1 - g^{n+1})(x)d\nu(x) = \int_E (g + g^2 + \ldots + g^{n+1})(x)d\mu(x).$$

By Monotone Convergence Theorem $\lim\limits_{n\to+\infty} \int_E (1-g^{n+1})(x)d\nu(x) = \nu(E \cap A) = \nu_a(E)$. Moreover the sequence $(g + g^2 + \ldots + g^{n+1})$ is increasing and converges to a function h and

$$\lim \int (g + g^2 + \ldots + g^{n+1})(x)d\mu(x) = \int h(x)d\mu(x).$$

We assume now that ν is finite and μ $\sigma-$finite. Let $(E_n)_n$ be a disjoint sequence of measurable sets such that $\mu(E_n) < +\infty$ and $X = \cup_{n=1}^{\infty} E_n$. The previous case proves that there exists a measurable function $h_n \in L^1(X, \mathscr{A} \cap E_n, \mu_{\restriction E_n})$ such that $\nu_{\restriction E_n} = h_n \mu_{\restriction E_n} + \nu_{s,n}$.

We define $h = \displaystyle\sum_{n=1}^{\infty} h_n \chi_{E_n}$, $\nu_s = \displaystyle\sum_{n=1}^{\infty} nu_{s,n}$ and the conclusion is valid because $h \in L^1(X, \mathscr{A}, \mu)$ since $\nu(X) < +\infty$.

We assume now that μ and ν are $\sigma-$finite. As in the second case there exist a disjoint sequence of measurable sets $(E_n)_n$ such that $\mu(E_n) < +\infty$, $\nu(E_n) < +\infty$ and $X = \cup_{n=1}^{\infty} E_n$. The previous case proves that for all $n \in \mathbb{N}$, there exists a measurable function $h_n \in L^1(X, \mathscr{A} \cap E_n, \mu_{\restriction E_n})$ such that $\nu_{\restriction E_n} = h_n \mu_{\restriction E_n} + \nu_{s,n}$. We take

$$h = \sum_{n=1}^{\infty} h_n \chi E_n, \ \nu_s = \sum_{n=1}^{\infty} \nu_{s,n}. \qquad \square$$

The function h obtained in this theorem is called the Radon-Nikodym derivative of ν with respect to μ, and it is usually denoted by $\dfrac{d\nu}{d\mu}$. The justification for this notation is that it satisfies familiar calculus properties.

Remark

Let $X = [0,1]$ and $f\colon [0,1] \longrightarrow [0,1]$ be a differentiable function whose derivative is bounded and nowhere zero. If λ is the Lebesgue measure on $[0,1]$, then $\nu(E) = \lambda(f^{-1}E)$ is a measure on $[0,1]$ which is absolutely continuous with respect to λ, and the Radon-Nikodym derivative $\dfrac{d\nu}{d\lambda}$ is $|f'(x)|$. This is deduced by the Change of Variable Theorem.

Remark

If we remove the assumption that the measure are $\sigma-$finite the conclusion of the Theorem (11) can be false. Let $X = [0, 1]$, λ the Lebesgue measure and ν the counting measure on X. $\lambda << \nu$ but there exit no function h such that $\lambda = h\nu$.