6 Inner Products Spaces

6.1 Inner Products

6.2 Angle and orthogonality in inner product spaces

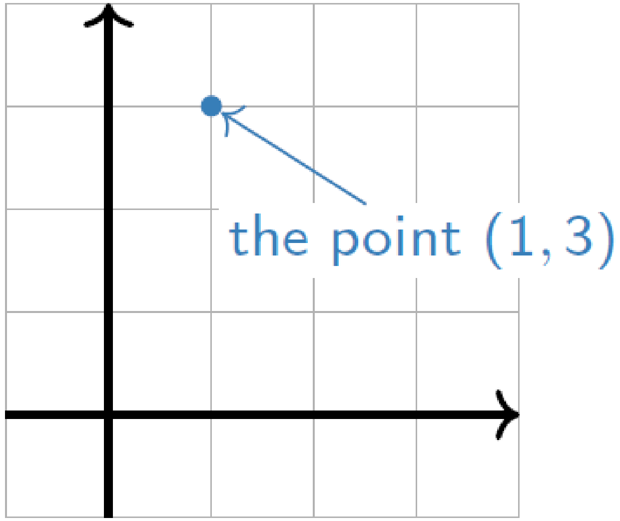
6.3 Gram-Schmit Process

6.1 Inner Products

Points and Vectors in R^n

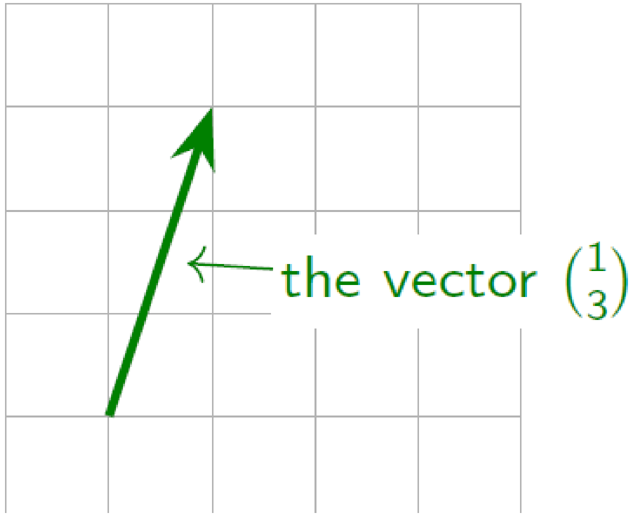
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).



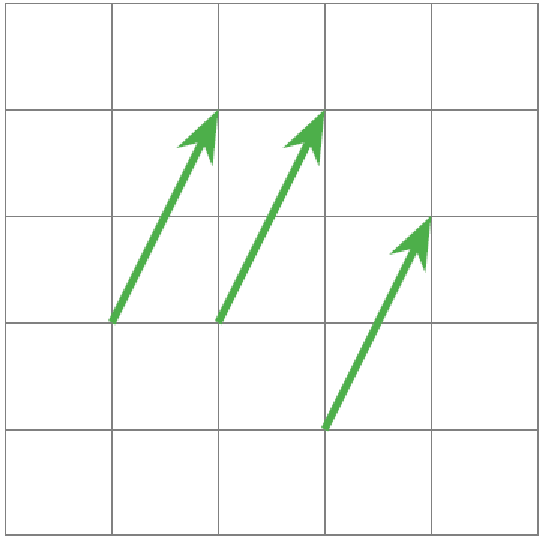
A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$



So why make the distinction?

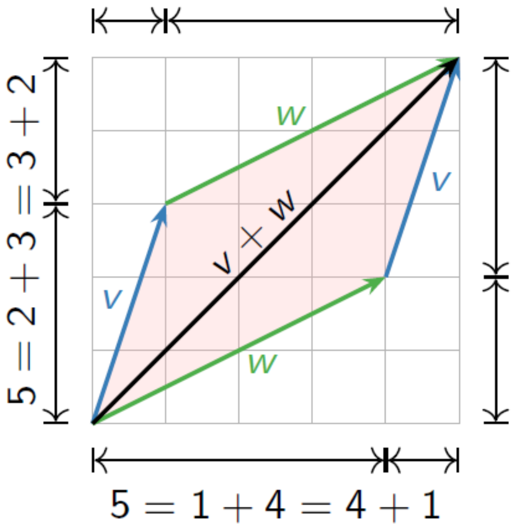
A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.

Vector Addition and Subtraction **Geometrically**

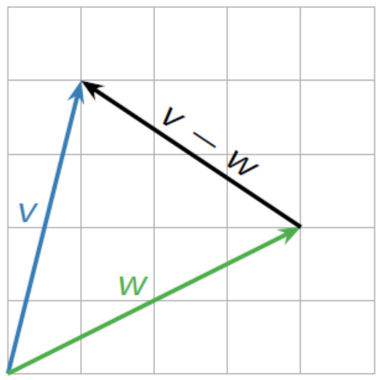


The parallelogram law for vector addition

Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v + w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights.



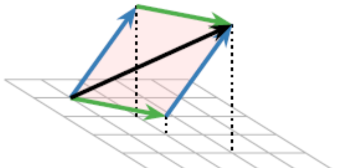
Vector subtraction

Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v. For example,

$$\begin{pmatrix}1\\4\end{pmatrix} - \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix}.$$

Why? If you add $v - w$ to w, you get v.

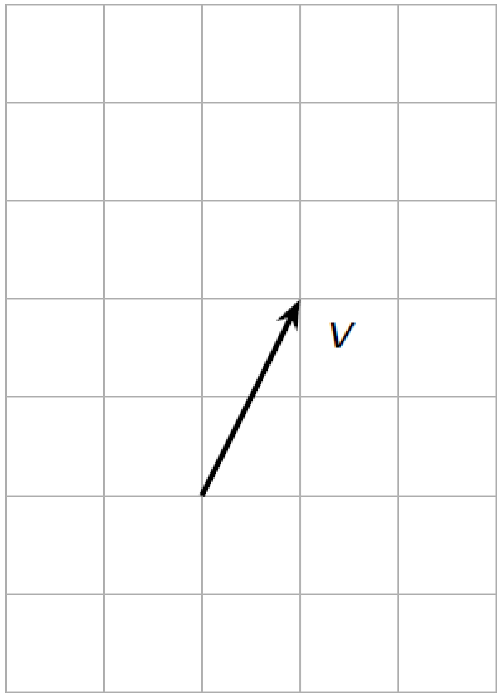
This works in higher dimensions too!



Scalar multiples of a vector

These have the same *direction* but a different *length*.

Some multiples of v.



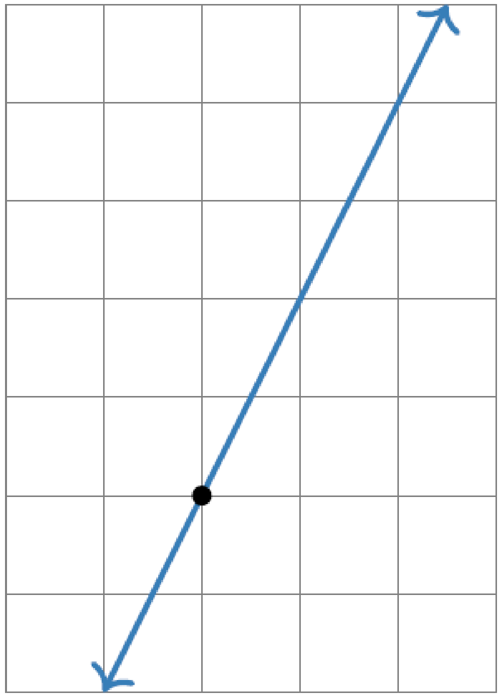
$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of v.



So the scalar multiples of v form a *line*.

Length and Dot Product in R^n

- Length:

The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{v_1^{\ 2} + v_2^{\ 2} + \cdots + v_n^{\ 2}} \quad (\|\mathbf{v}\| \text{ is a real number})$$

- Note: The length of a vector is also called its norm

- Properties of length (or norm)

(1) $\|\mathbf{v}\| \geq 0$

(2) $\|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a unit vector

(3) $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

(4) $\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$

Example:

(a) In R^5, the length of $\mathbf{v} = (0, -2, 1, 4, -2)$ is given by

$$\|\mathbf{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

(b) In R^3, the length of $\mathbf{v} = (\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1$$

(If the length of **v** is 1, then **v** is a unit vector)

Example:

A standard unit vector in R^n: only one component of the vector is 1 and the others are 0 (thus the length of this vector must be 1)

$$R^2 : \{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0),(0,1)\}$$

$$R^3 : \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$$

$$R^n : \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0),(0,1,\cdots,0),\cdots,(0,0,\cdots,1)\}$$

Notes: Two nonzero vectors are parallel if $\mathbf{u} = c\mathbf{v}$

(1) $c > 0 \;\Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \;\Rightarrow$ **u** and **v** have the opposite directions

Theorem: Length of a scalar multiple

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Proof:

$$\mathbf{v} = (v_1\,,\, v_2\,,\cdots,\, v_n)$$

$$\Rightarrow c\mathbf{v} = (cv_1\,,\, cv_2\,,\cdots,\, cv_n)$$

$$\|c\mathbf{v}\| = \|\,(cv_1\,,\, cv_2\,,\cdots,\, cv_n)\,\|$$

$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$

$$= |c|\,\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$

$$= |c|\,\|\mathbf{v}\|$$

Theorem: How to find the unit vector in the direction of $\mathbf{v}$

If $\mathbf{v}$ is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

has length 1 and has the same direction as $\mathbf{v}$. This vector $\mathbf{u}$ is called the unit vector in the direction of $\mathbf{v}$

Proof:

$\mathbf{v}$ is nonzero $\Rightarrow \|\mathbf{v}\| > 0 \Rightarrow \dfrac{1}{\|\mathbf{v}\|} > 0$

If $\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ ($\mathbf{u}$ has the same direction as $\mathbf{v}$)

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| \overset{\|c\mathbf{v}\| = |c|\|\mathbf{v}\|}{=} \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \text{ (}\mathbf{u}\text{ has length 1)}$$

Notes:

(1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of $\mathbf{v}$

(2) The process of finding the unit vector in the direction of $\mathbf{v}$ is called **normalizing** the vector $\mathbf{v}$

Example: Finding a unit vector

Find the unit vector in the direction of **v** = (3, –1, 2), and verify that this vector has length 1

Solution:

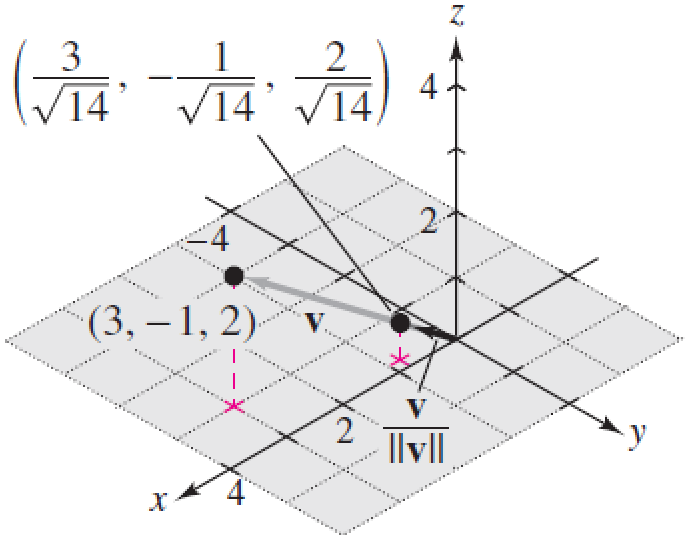
$$\mathbf{v}=(3,-1,2)\Rightarrow\|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3,-1,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3,-1,2)$$

$$=\left(\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}},\frac{2}{\sqrt{14}}\right)$$

$$\because\ \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

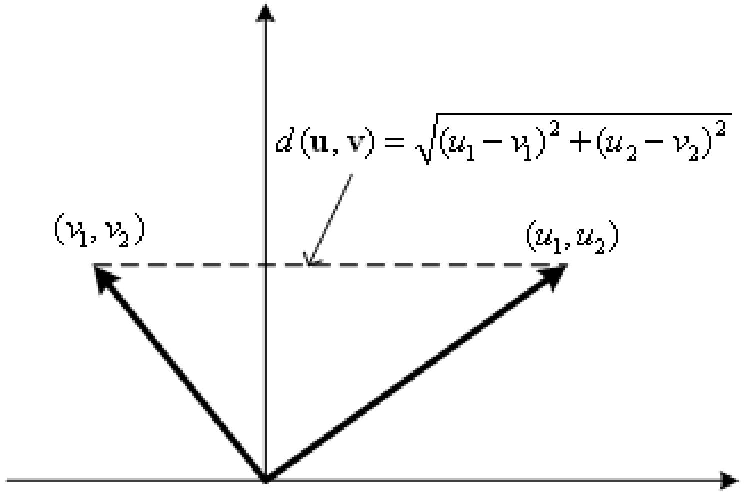
$$\therefore\ \frac{\mathbf{v}}{\|\mathbf{v}\|}\ \text{is a unit vector}$$



- Distance between two vectors:

  The distance between two vectors **u** and **v** in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$



- Properties of distance

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ (commutative property of the distance function)

Definition: Dot product in R^n

The dot product of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$

returns a scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \quad (\mathbf{u} \cdot \mathbf{v} \text{ is a real number})$$

(The dot product is defined as the sum of component-by-component multiplications)

Example: Finding the dot product of two vectors

The dot product of $\mathbf{u} = (1, 2, 0, -3)$ and $\mathbf{v} = (3, -2, 4, 2)$ is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

Theorem: Properties of the dot product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n and c is a scalar, then the following properties are true

(1) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutative property of the dot product)

(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributive property of the dot product over vector addition)

(3) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$ (associative property of the scalar multiplication and the dot product)

(4) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \Rightarrow \mathbf{v} \cdot \mathbf{v} \geq 0$

(5) $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (straightforwardly true according to (4))

※ The proofs of the above properties simply follow the definition of the dot product in R^n

Euclidean *n*-space:

When R^n is combined with the standard operations of **vector addition**, **scalar multiplication**, **vector length**, and **dot product**, the resulting vector space is called **Euclidean *n*-space**

Example: Computations With Dot Products

Compute the following for:

$$\mathbf{u} = (2\,,-2)\,,\ \mathbf{v} = (5\,,8),\ \mathbf{w} = (-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Solution:

(a) $\mathbf{u}\cdot\mathbf{v} = (2)(5)+(-2)(8) = -6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = -6\mathbf{w} = -6(-4\,,3) = (24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v}) = 2(\mathbf{u}\cdot\mathbf{v}) = 2(-6) = -12$

(d) $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = (-4)(-4)+(3)(3) = 25$

(e) $\mathbf{v}-2\mathbf{w} = (5-(-8)\,,8-6) = (13\,,2)$

$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w}) = (2)(13)+(-2)(2) = 26-4 = 22$

Theorem: The Cauchy-Schwarz inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\| \quad (\,|\mathbf{u}\cdot\mathbf{v}|\text{ denotes the absolute value of } \mathbf{u}\cdot\mathbf{v})$$

Proof: A more general proof will be given later.

Example: Verifying Cauchy-Schwarz inequality

Verify the Cauchy-Schwarz inequality for $\mathbf{u} = (1, -1, 3)$ and $\mathbf{v} = (2, 0, -1)$.

Solution:

$$\mathbf{u}\cdot\mathbf{v} = -1,\ \ \mathbf{u}\cdot\mathbf{u} = 11,\ \ \mathbf{v}\cdot\mathbf{v} = 5$$

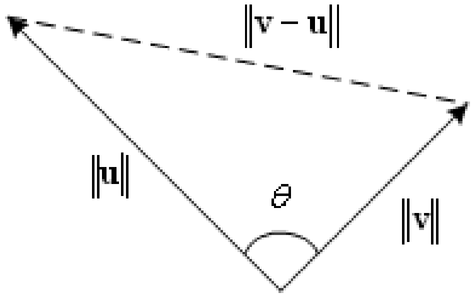
$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

- Dot product and the angle between two vectors

To find the angle $\theta\,(0 \le \theta \le \pi)$ between two nonzero vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in R^2, the Law of Cosines can be applied to the following triangle to obtain



$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\|\mathbf{u}\|\cos\theta$$

$$\because \|\mathbf{v}-\mathbf{u}\|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2$$

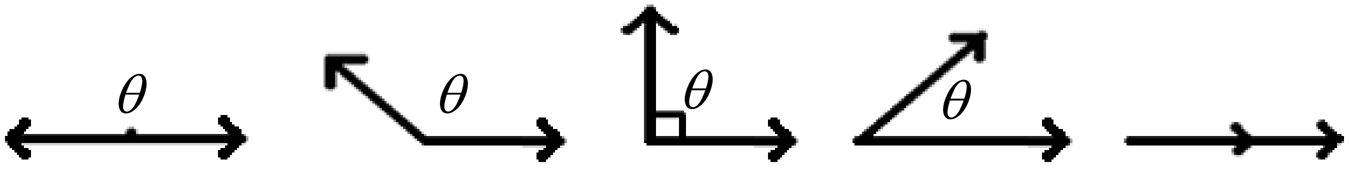
$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2$$

$$\therefore \cos\theta = \frac{u_1 v_1 + u_2 v_2}{\|\mathbf{v}\|\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}\|}$$

- The angle between two nonzero vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$



Opposite direction	$\mathbf{u}\cdot\mathbf{v} < \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} = \mathbf{0}$	$\mathbf{u}\cdot\mathbf{v} > \mathbf{0}$	Same direction
$\theta = \pi$	$\frac{\pi}{2} < \theta < \pi$	$\theta = \frac{\pi}{2}$	$0 < \theta < \frac{\pi}{2}$	$\theta = 0$
$\cos\theta = -1$	$\cos\theta < 0$	$\cos\theta = 0$	$\cos\theta > 0$	$\cos\theta = 1$

Note:

The angle between the zero vector and another vector is not defined (since the denominator cannot be zero)

Example: Finding the angle between two vectors

$$\mathbf{u} = (-4, 0, 2, -2) \quad \mathbf{v} = (2, 0, -1, 1)$$

Solution:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{2^2 + (0)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{u} \cdot \mathbf{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1$$

$$\Rightarrow \theta = \pi \qquad \therefore \ \mathbf{u} \text{ and } \mathbf{v} \text{ have opposite directions}$$

(In fact, **u** = –2**v** and as discussed, **u** and **v** are parallel and with different directions)

- Orthogonal vectors:

Two vectors **u** and **v** in R^n are orthogonal (perpendicular) if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

The vector **0** is said to be orthogonal to every vector

Example: Finding orthogonal vectors

Determine all vectors in R^n that are orthogonal to $\mathbf{u} = (4, 2)$

Solution:

$$\mathbf{u} = (4\,,2) \quad \text{Let} \quad \mathbf{v} = (v_1\,, v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v} = (4\,,2)\cdot(v_1\,, v_2)$$

$$= 4v_1 + 2v_2$$

$$= 0$$

$$\Rightarrow \quad v_1 = \frac{-t}{2} \quad , \quad v_2 = t$$

$$\therefore \quad \mathbf{v} = \left(\frac{-t}{2}, t\right), \; t \in R$$

Theorem: The Triangle Inequality

If **u** and **v** are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Proof:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \quad (c \le |c|)$$

$$\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \quad \text{(Cauchy-Schwarz inequality)}$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

(The geometric representation of the triangle inequality: for any triangle, the sum of the lengths of any two sides is larger than the length of the third side (see the next slide))

Note:

Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction (in this situation, $\cos\theta = 1$ and thus $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\| \ge 0$)

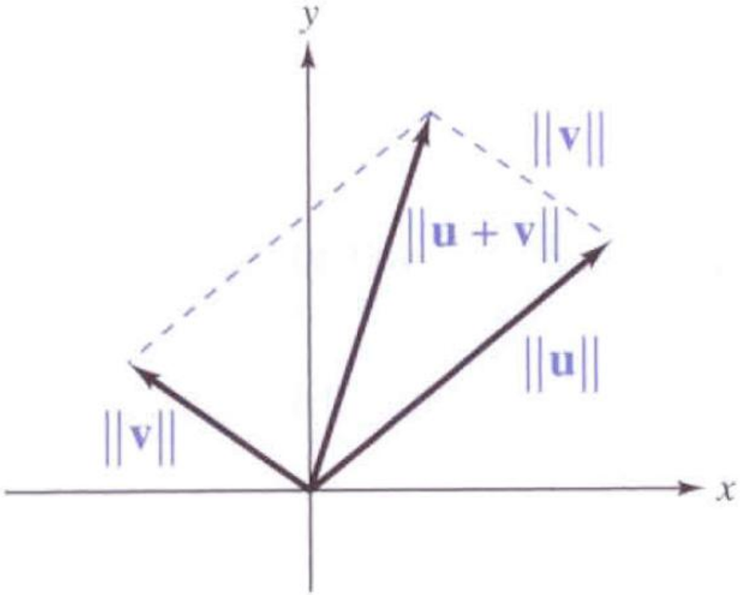
Theorem: The Pythagorean theorem

If **u** and **v** are vectors in R^n, then **u** and **v** are orthogonal if and only if

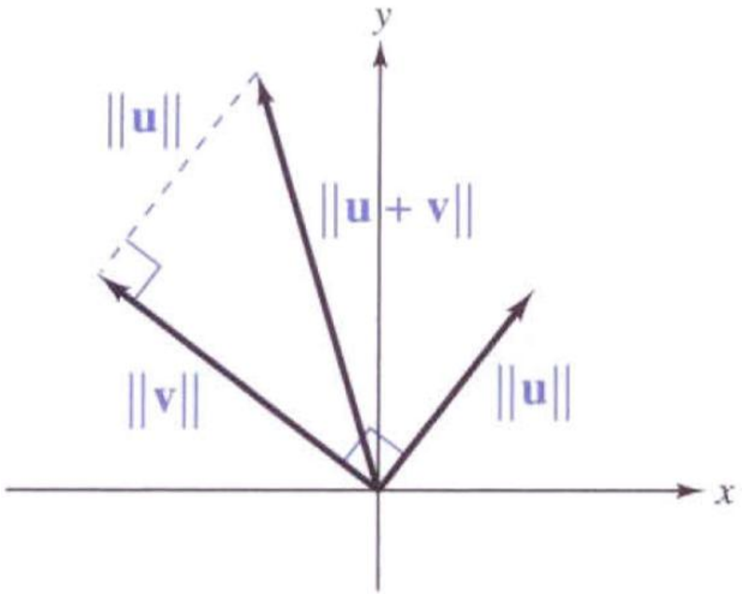
$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

(This is because $\mathbf{u}\cdot\mathbf{v} = 0$ in the proof for the last Theorem)

※ The geometric meaning: for any right triangle, the sum of the squares of the lengths of two legs equals the square of the length of the hypotenuse.



$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$



$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Definition: Inner Product Space

Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}\,,\, \mathbf{v}\rangle$ with each pair of vectors **u** and **v** and satisfies the following axioms (abstraction definition from the properties of dot product)

(1) $\langle \mathbf{u}\,,\, \mathbf{v}\rangle = \langle \mathbf{v}\,,\, \mathbf{u}\rangle$ (SYMMETRY)

(2) $\langle \mathbf{u}\,,\, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}\,,\, \mathbf{v}\rangle + \langle \mathbf{u}\,,\, \mathbf{w}\rangle$ (ADDITIVITY)

(3) $c\,\langle \mathbf{u}\,,\, \mathbf{v}\rangle = \langle c\mathbf{u}\,,\, \mathbf{v}\rangle$ (HOMOGENEITY)

(4) $\langle \mathbf{v}\,,\, \mathbf{v}\rangle \geq 0$ (POSITIVITY)

(5) $\langle \mathbf{v}\,,\, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ (DEFINITENESS)

Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for a vector space } V$$

Note:

A vector space V with an inner product is called an inner product space

Vector space: $(V,\ +,\ \cdot)$

Inner product space: $(V,\ +,\ \cdot\ ,\ \ <,\ >)$

Example: The Euclidean inner product for R^n

Show that the dot product in R^n satisfies the four axioms of an inner product.

Solution:

$$\mathbf{u} = (u_1\ , u_2\ , \cdots , u_n) \quad , \quad \mathbf{v} = (v_1\ , v_2\ , \cdots , v_n)$$

$$\langle \mathbf{u}\ , \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By a previous Theorem, this dot product satisfies the required four axioms. Thus, the dot product can be a sort of inner product in R^n

Example: A different inner product for R^n

Show that the following function defines an inner product on R^2. Given $\mathbf{u} = (u_1 , u_2)$ and $\mathbf{v} = (v_1 , v_2)$,

$$\langle \mathbf{u} , \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Solution:

(1) $\langle \mathbf{u} , \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \langle \mathbf{v} , \mathbf{u} \rangle$

(2) $\mathbf{w} = (w_1 , w_2)$

$$\begin{aligned} \Rightarrow \langle \mathbf{u} , \mathbf{v} + \mathbf{w} \rangle &= u_1 (v_1 + w_1) + 2u_2 (v_2 + w_2) \\ &= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2 \\ &= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2) \\ &= \langle \mathbf{u} , \mathbf{v} \rangle + \langle \mathbf{u} , \mathbf{w} \rangle \end{aligned}$$

$$(3)\quad c\,\langle \mathbf{u}\,,\,\mathbf{v}\rangle = c(u_1v_1+2u_2v_2)=(cu_1)v_1+2(cu_2)v_2=\langle c\mathbf{u}\,,\,\mathbf{v}\rangle$$

$$(4)\quad \langle \mathbf{v}\,,\,\mathbf{v}\rangle = v_1{}^2+2v_2{}^2\geq 0$$

$$(5)\quad \langle \mathbf{v}\,,\,\mathbf{v}\rangle = 0 \Rightarrow v_1{}^2+2v_2{}^2=0\Rightarrow v_1=v_2=0\quad (\mathbf{v}=\mathbf{0})$$

Note: This example can be generalized such that

$< \mathbf{u}\,,\,\mathbf{v} >= c_1u_1v_1+c_2u_2v_2+\cdots+c_nu_nv_n$, where all $c_i > 0$

can be an inner product on R^n

Example: A function that is not an inner product

Show that the following function is not an inner product on R^3

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - 2u_2v_2 + u_3v_3$$

Solution:

Let $\mathbf{v} = (1, 2, 1)$

Then $\langle \mathbf{v}, \mathbf{v} \rangle = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied

Thus this function is not an inner product on R^3

Example (Inner Product on $M_{2\times 2}$):

Inner Product Definition: For $A, B \in M_{2\times 2}$ = all 2×2 matrices

$$\langle A, B\rangle = \mathrm{trace}(AB^T)$$

Concrete Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. **Compute** $\langle A, B\rangle$:

$$AB^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$\langle A, B\rangle = \mathrm{trace}(AB^T) = 2 + 3 = 5$$

Show that this satisfies all inner product axioms:

1. **Symmetry**: $\langle A, B \rangle = \langle B, A \rangle$
2. **Additivity**: $\langle A_1 + A_2, B \rangle = \langle A_1, B \rangle + \langle A_2, B \rangle$
3. **Homogeneity**: $\langle cA, B \rangle = c\langle A, B \rangle$
4. **Positivity**: $\langle A, A \rangle \geq 0$ for all A
5. **Definiteness**: $\langle A, A \rangle = 0 \iff A = 0$

Axiom 1: Symmetry $\langle A, B\rangle = \langle B, A\rangle$

$$\langle A, B\rangle = \text{trace}(AB^T) = \text{trace}(B^TA) = \text{trace}((B^TA)^T) = \text{trace}(A^TB) = \text{trace}(BA^T) = \langle B, A\rangle$$

Axiom 2: Additivity $\langle A_1 + A_2, B\rangle = \langle A_1, B\rangle + \langle A_2, B\rangle$

$$\langle A_1 + A_2, B\rangle = \text{trace}((A_1 + A_2)B^T) = \text{trace}(A_1B^T + A_2B^T) = \text{trace}(A_1B^T) + \text{trace}(A_2B^T)$$

$$= \langle A_1, B\rangle + \langle A_2, B\rangle$$

Axiom 3: Homogeneity $\langle cA, B\rangle = c\langle A, B\rangle$ for all scalars c

$$\langle cA, B\rangle = \text{trace}((cA)B^T) = \text{trace}(c(AB^T)) = c \cdot \text{trace}(AB^T) = c\langle A, B\rangle$$

Axiom 4: Positivity $\langle A, A\rangle \geq 0$ for all $A \in M_{2\times 2}$

$$\langle A, A\rangle = \text{trace}(AA^T)$$

$$AA^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{12}^2 & * \\ * & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

$$\text{trace}(AA^T) = (a_{11}^2 + a_{12}^2) + (a_{21}^2 + a_{22}^2) = \sum_{i,j} a_{ij}^2 \geq 0$$

Axiom 5: Definiteness $\langle A, A \rangle = 0 \iff A = 0$ (zero matrix)

$$\langle A, A \rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$$

Since each term is non-negative, this requires:

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{21} = 0, \quad a_{22} = 0$$

Therefore $A = 0$ (the zero matrix)

Example (Inner Product on P$_2$):

Inner Product Definition: For $p, q \in P_2$, polynomials of degree ≤ 2.

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

Example: If $p(x) = x^2 + 1$ and $q(x) = 2x + 3$, then

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2) = (1)(3) + (2)(5) + (5)(7) = 3 + 10 + 35 = 48$$

Exercise: Show that this satisfies all inner product axioms.

Theorem: Properties of inner products

Let **u**, **v**, and **w** be vectors in an inner product space V, and let c be any real number

(1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(2) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

※ To prove these properties, you can use only the four axioms for defining an inner product

Proof:

(1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0\mathbf{u}, \mathbf{v} \rangle \overset{(3)}{=} 0 \langle \mathbf{u}, \mathbf{v} \rangle = 0$

(2) $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle \overset{(1)}{=} \langle \mathbf{w}, \mathbf{u}+\mathbf{v} \rangle \overset{(2)}{=} \langle \mathbf{w}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \overset{(1)}{=} \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(3) $< \mathbf{u}, c\mathbf{v} > \overset{(1)}{=} < c\mathbf{v}, \mathbf{u} > \overset{(3)}{=} c < \mathbf{v}, \mathbf{u} > \overset{(1)}{=} c < \mathbf{u}, \mathbf{v} >$

Definition of **length**, **distance**, **angle**, **orthogonal**, and **normalizing** for general inner product spaces is a generalization to those based on the dot product in Euclidean *n*-space

- **Length** of **u**:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- **Distance** between **u** and **v**:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- **Angle** between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \ 0 \le \theta \le \pi$$

- **Orthogonal**: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

- **Normalizing** vectors

(1) If $\|\mathbf{v}\| = 1$, then **v** is called a unit vector

(Note that $\|\mathbf{v}\|$ is defined as $\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$)

(2) $\mathbf{v} \neq \mathbf{0} \xrightarrow{\text{Normalizing}} \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

(if **v** is not a zero vector)

Example: An inner product in the polynomial space

For $p = a_0 + a_1 x + \cdots + a_n x^n$ and $q = b_0 + b_1 x + \cdots + b_n x^n$,

and $\langle p\,, q\rangle \equiv a_0 b_0 + a_1 b_1 + \cdots + a_n b_n$ is an inner product

Let $p(x) = 1 - 2x^2, q(x) = 4 - 2x + x^2$ be polynomials in P_2

(a) $\langle p\,, q\rangle = ?$ (b) $\|q\| = ?$ (c) $d(p\,, q) = ?$

Solution:

(a) $\langle p\,, q\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$

(b) $\|q\| = \sqrt{\langle q\,, q\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$

(c) $\because\ p - q = -3 + 2x - 3x^2$

$$\therefore d(p\,, q) = \|p - q\| = \sqrt{\langle p - q, p - q\rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm: (the same as the properties for the dot product in R^n)

  (1) $\|\mathbf{u}\| \geq 0$

  (2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

  (3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance: (the same as the properties for the dot product in R^n)

  (1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

  (2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

  (3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

Exercise:

True or False?

If u is orthogonal to every vector of a subspace W, then $u = 0$.

A)True.
B)False.

Multiple Choice

Exercise:

True or False?

If $u \in W$ is orthogonal to every vector of a subspace W, then $u = 0$.

A)True.

B)False.

Multiple Choice

Theorem:

Let **u** and **v** be vectors in an inner product space V

(1) **Cauchy-Schwarz inequality**:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

(2) **Triangle inequality**:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

(3) **Pythagorean theorem**:

u and **v** are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

(4) **Parallelogram Identity**:

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\,\|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|^2$$

Proof: Cauchy-Schwarz inequality

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Let $Q(t)$ be the polynomial

$$Q(t) = \|u + tv\|^2 = \|u\|^2 + 2t\langle u, v\rangle + t^2\|v\|^2.$$

Since $Q(t) \geq 0$ for all $t \in \mathbb{R}$, then the discriminant of $Q(t)$ is non positive. Then

$$\langle u, v\rangle^2 \leq \|u\|^2\|v\|^2.$$

If $|\langle u, v\rangle| = \|u\|\|v\|$, this mean that the discriminant of $Q(t)$ is zero. Then the equation $Q(t) = 0$ has a solution. This means that the vectors u, v are linearly dependent.

Proof of Pythagorean Theorem

Given: $u, v \in V$ with $\langle u, v \rangle = 0$. **To prove:** $\|u + v\|^2 = \|u\|^2 + \|v\|^2$:

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u + v \rangle + \langle v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$

$$= \langle u, u \rangle + 0 + 0 + \langle v, v \rangle = \langle u, u \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2$$

Proof of Parallelogram Identity: $||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2\,||\mathbf{u}||^2 + 2\,||\mathbf{v}||^2$

$$||u+v||^2 + ||u-v||^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \langle u,u\rangle + \langle u,-v\rangle + \langle -v,u\rangle + \langle -v,-v\rangle$$

$$= 2\langle u,u\rangle + 2\langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle - \langle u,v\rangle - \langle v,u\rangle$$

$$= 2||u||^2 + 2||v||^2$$

Orthonormal Bases: Gram-Schmidt Process

- Orthogonal set:

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \text{ for } i \neq j$$

- Orthonormal set:

An orthogonal set in which each vector is a unit vector is called orthonormal set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$\begin{cases} \text{For } i = j, \ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \|\mathbf{v}_i\|^2 = 1 \\ \text{For } i \neq j, \ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \end{cases}$$

Note:

- If S is also a basis, then it is called an orthogonal basis or an orthonormal basis
- The standard basis for R^n is orthonormal. For example,

$$S = \{(1,0,0),(0,1,0),(0,0,1)\}$$

is an orthonormal basis for R^3

- This section identifies some advantages of orthonormal bases, and develops a procedure for constructing such bases, known as Gram-Schmidt orthonormalization process

Example: A nonstandard orthonormal basis for R^3

Show that the following set is an orthonormal basis

$$S = \left\{ \overset{\mathbf{v}_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)}, \quad \overset{\mathbf{v}_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right)}, \quad \overset{\mathbf{v}_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right)} \right\}$$

Solution:

First, note that the three vectors are mutually orthogonal

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -\tfrac{1}{6} + \tfrac{1}{6} + 0 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Second, note that each vector is of length 1

$$\|\mathbf{v}_1\| = \sqrt{\mathbf{v}_1 \cdot \mathbf{v}_1} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = 1$$

$$\|\mathbf{v}_2\| = \sqrt{\mathbf{v}_2 \cdot \mathbf{v}_2} = \sqrt{\tfrac{2}{36} + \tfrac{2}{36} + \tfrac{8}{9}} = 1$$

$$\|\mathbf{v}_3\| = \sqrt{\mathbf{v}_3 \cdot \mathbf{v}_3} = \sqrt{\tfrac{4}{9} + \tfrac{4}{9} + \tfrac{1}{9}} = 1$$

Thus, S is an orthonormal set

Because these three vectors are linearly independent (you can check by solving $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$) in R^3 (of dimension 3), by Theorem (given a vector space with dimension n, then n linearly independent vectors can form a basis for this vector space), these three linearly independent vectors form a basis for R^3.

$\Rightarrow$ S is a (nonstandard) orthonormal basis for R^3

Example: An orthonormal basis for $P_2(x)$

In $P_2(x)$, with the inner product $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$,

the standard basis $B = \{1, x, x^2\}$ is orthonormal

Solution:

$$\mathbf{v}_1 = 1 + 0x + 0x^2, \qquad \mathbf{v}_2 = 0 + x + 0x^2, \qquad \mathbf{v}_3 = 0 + 0x + x^2,$$

Then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = (1)(0) + (0)(1) + (0)(0) = 0$$
$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (1)(0) + (0)(0) + (0)(1) = 0$$
$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = (0)(0) + (1)(0) + (0)(1) = 0$$

$$\|\mathbf{v}_1\| = \sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} = \sqrt{(1)(1) + (0)(0) + (0)(0)} = 1$$
$$\|\mathbf{v}_2\| = \sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} = \sqrt{(0)(0) + (1)(1) + (0)(0)} = 1$$
$$\|\mathbf{v}_3\| = \sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle} = \sqrt{(0)(0) + (0)(0) + (1)(1)} = 1$$

Theorem: Orthogonal sets are linearly independent

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an **orthogonal** set of **nonzero** vectors in an inner product space V, then S is **linearly independent**

Proof:

S is an orthogonal set of nonzero vectors,

i.e., $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$, and $\langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$

For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ (If there is only the trivial solution for c_i's, i.e., all c_i's are 0, S is linearly independent)

$$\Rightarrow \quad \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle = 0 \qquad \forall i$$

$$\Rightarrow \quad c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle$$

$= c_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0$ (because S is an orthogonal set of nonzero vectors)

$\because \langle \mathbf{v}_i, \mathbf{v}_i \rangle \neq 0 \quad \Rightarrow c_i = 0 \quad \forall i \quad \therefore S$ is linearly independent

Corollary:

If V is an inner product space with dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Proof:

1. By a Theorem , if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set nonzero vectors, then S is linearly independent.
2. According to a previous Theorem, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of n vectors in V (with dimension n), then S is a basis for V .

※ Based on the above two arguments, it is straightforward to derive the above corollary.

Exercise:

In a vector space of dimension $n \geq 1$ the largest set of distinct orthogonal vectors has:

$A.\ n-1$ vectors

B. n vectors

C. $n+1$ vectors

D. infinite number of vectors

★ Multiple Choice

Example: Using orthogonality to test for a basis

Show that the following set is a basis for R^4

$$S=\{\overset{\mathbf{v}_1}{(2,3,2,-2)},\overset{\mathbf{v}_2}{(1,0,0,1)},\overset{\mathbf{v}_3}{(-1,0,2,1)},\overset{\mathbf{v}_4}{(-1,2,-1,1)}\}$$

Solution:

$\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4$: nonzero vectors

$$\mathbf{v}_1\cdot\mathbf{v}_2=2+0+0-2=0 \qquad \mathbf{v}_2\cdot\mathbf{v}_3=-1+0+0+1=0$$

$$\mathbf{v}_1\cdot\mathbf{v}_3=-2+0+4-2=0 \qquad \mathbf{v}_2\cdot\mathbf{v}_4=-1+0+0+1=0$$

$$\mathbf{v}_1\cdot\mathbf{v}_4=-2+6-2-2=0 \qquad \mathbf{v}_3\cdot\mathbf{v}_4=1+0-2+1=0$$

$\Rightarrow S$ is orthogonal

$\Rightarrow S$ is a basis for R^4 (by Corollary)

※ This shows an advantage of introducing the concept of orthogonal vectors, i.e., it is not necessary to solve linear systems to test whether S is a basis if S is a set of orthogonal vectors.

Theorem: Coordinates relative to an orthonormal basis

If $B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the unique coordinate representation of a vector $\mathbf{w}$ with respect to B is

$$\mathbf{w}=\langle \mathbf{w}, \mathbf{v}_1\rangle \mathbf{v}_1+\langle \mathbf{w}, \mathbf{v}_2\rangle \mathbf{v}_2+\cdots+\langle \mathbf{w}, \mathbf{v}_n\rangle \mathbf{v}_n$$

※ The above theorem tells us that it is easy to derive the coordinate representation of a vector relative to an orthonormal basis, which is another advantage of using orthonormal bases

Proof:

$B=\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V

$\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+\cdots+k_n\mathbf{v}_n \in V$ (unique representation from Thm. 4.9)

Since $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$, then

$$\begin{aligned}\left\langle \mathbf{w},\, \mathbf{v}_i \right\rangle &= \left\langle (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n),\, \mathbf{v}_i \right\rangle \\ &= k_1 \left\langle \mathbf{v}_1,\, \mathbf{v}_i \right\rangle + \cdots + k_i \left\langle \mathbf{v}_i,\, \mathbf{v}_i \right\rangle + \cdots + k_n \left\langle \mathbf{v}_n,\, \mathbf{v}_i \right\rangle \\ &= k_i \qquad \text{for } i = 1 \text{ to } n\end{aligned}$$

$$\Rightarrow \mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Note:

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is an orthonormal basis for V and $\mathbf{w} \in V$,

Then the corresponding coordinate matrix of **w** relative to B is

$$[\mathbf{w}]_B = \begin{bmatrix} \langle \mathbf{w}, \mathbf{v}_1 \rangle \\ \langle \mathbf{w}, \mathbf{v}_2 \rangle \\ \vdots \\ \langle \mathbf{w}, \mathbf{v}_n \rangle \end{bmatrix}$$

Example: Coordinates relative to an orthonormal basis

For $\mathbf{w} = (5, -5, 2)$, find its coordinates relative to the standard basis for R^3

$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5\,,-5\,,2) \cdot (1\,,0\,,0) = 5$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5,-5\,,2) \cdot (0\,,1\,,0) = -5$$

$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5\,,-5\,,2) \cdot (0\,,0\,,1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} 5 \\ -5 \\ 2 \end{bmatrix}$$

※ In fact, it is not necessary to use Thm. to find the coordinates relative to the standard basis, because we know that the coordinates of a vector relative to the standard basis are the same as the components of that vector.

※ The advantage of the orthonormal basis emerges when we try to find the coordinate matrix of a vector relative to a nonstandard orthonormal basis (see the next slide).

Example: Representing vectors relative to an orthonormal basis

Find the coordinates of $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for R^3

$$B = \{\overset{\mathbf{v}_1}{(\tfrac{3}{5}, \tfrac{4}{5}, 0)}, \overset{\mathbf{v}_2}{(-\tfrac{4}{5}, \tfrac{3}{5}, 0)}, \overset{\mathbf{v}_3}{(0, 0, 1)}\}$$

Solution:

$$\langle \mathbf{w}, \mathbf{v}_1 \rangle = \mathbf{w} \cdot \mathbf{v}_1 = (5, -5, 2) \cdot (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$

$$\langle \mathbf{w}, \mathbf{v}_2 \rangle = \mathbf{w} \cdot \mathbf{v}_2 = (5, -5, 2) \cdot (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7$$

$$\langle \mathbf{w}, \mathbf{v}_3 \rangle = \mathbf{w} \cdot \mathbf{v}_3 = (5, -5, 2) \cdot (0, 0, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

Exercise:

True or False?

If $u \in W$ is orthogonal to every vector of W, then $u = 0$.

A) True.

B) False.

Multiple Choice

Gram-Schmidt orthonormalization process:

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for an inner product space V

Let $\mathbf{w}_1 = \mathbf{v}_1$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \sum_{i=1}^{n-1} \frac{\langle \mathbf{v}_n, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle}\mathbf{w}_i$$

$\Rightarrow B' = \{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n\}$ is an orthogonal basis

$\Rightarrow B'' = \{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \cdots, \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}\}$ is an orthonormal basis

Example: Applying the Gram-Schmidt process

Apply the Gram-Schmidt process to the following basis for R^3

$$B = \{\underset{\mathbf{v}_1}{(1,1,0)},\quad \underset{\mathbf{v}_2}{(1,2,0)},\quad \underset{\mathbf{v}_3}{(0,1,2)}\}$$

Solution:

$$\mathbf{w}_1 = \mathbf{v}_1 = (1,1,0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2},\frac{1}{2},0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2$$

$$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}(-\frac{1}{2},\frac{1}{2},0) = (0,0,2)$$

Orthogonal basis

$$\Rightarrow B' = \{\mathbf{w}_1 , \mathbf{w}_2 , \mathbf{w}_3\} = \{(1, 1, 0), (\frac{-1}{2}, \frac{1}{2}, 0), (0, 0, 2)\}$$

Orthonormal basis

$$\Rightarrow B'' = \{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}\} = \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (0, 0, 1)\}$$

- Alternative form of the Gram-Schmidt orthonormalization process:

$B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for an inner product space V

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \text{ where } \mathbf{w}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}, \text{ where } \mathbf{w}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2$$

$$\vdots$$

$$\mathbf{u}_n = \frac{\mathbf{w}_n}{\|\mathbf{w}_n\|}, \text{ where } \mathbf{w}_n = \mathbf{v}_n - \sum_{i=1}^{n-1} \langle \mathbf{v}_n, \mathbf{u}_i \rangle \mathbf{u}_i$$

$\Rightarrow \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is an orthonormal basis for V