Math 204
Differential Equations
Linear Differential Equations with Constant Coefficients

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1. Homogeneous Linear Differential Equations with Constant Coefficients
   - Second Order Homogeneous Linear Differential Equation (LDE) with Constant Coefficients
   - Characteristic Equation

2. Nonhomogeneous Linear Differential Equations with Constant Coefficients
   - Undetermined Coefficient Method

Homogeneous Linear Differential Equations with Constant Coefficients

Homogeneous Linear Differential Equations with Constant Coefficients

The homogeneous linear differential equations with constant coefficients has the general form

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = 0,$$

where $a_n, \dots, a_1, a_0$ are real constants, and $a_n \neq 0$.

Second Order Homogeneous Linear Differential equation with Constant Coefficients

To solve second order linear differential equations with constant coefficients

$$a\,y'' + b\,y' + cy = 0,$$

we assume

$$y = e^{mx}$$

and we get

$$a\,m^2 + b\,m + c = 0$$

this is called **characteristic equation**

Characteristic Equation

There are three cases for the roots of the characteristic equation

$$a\,m^2 + b\,m + c = 0$$

1. Two distinct real roots m_1, m_2, then the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

2. Repeated real root m_1, then the general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

3. Conjugate complex roots $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$, then the general solution is

$$y = c_1\, e^{\alpha x}\, \cos\beta x + c_2\, e^{\alpha x}\, \sin\beta x$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y'' - 5y' + 6y = 0$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y'' - 5y' + 6y = 0$$

Solution

$$y = c_1 e^{2x} + c_2 e^{3x}$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y'' + 2y' = 0$$

Example

Solve the differential equation

$$y'' + 2y' = 0$$

Solution

$$y = c_1 + c_2 \ e^{-2x}$$

Example

Solve the differential equation

$$y'' - 10y' + 25y = 0$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y'' - 10y' + 25y = 0$$

Solution

$$y = c_1 e^{5x} + c_2\, x\, e^{5x}$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y'' - 6y' + 13y = 0$$

Example

Solve the differential equation

$$y'' - 6y' + 13y = 0$$

Solution

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

Higher Order Homogeneous LDE with Constant Coefficients

The characteristic equation of

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = 0,$$

is

$$a_n\, m^n + a_{n-1}\, m^{n-1} + \cdots + a_1\, m + a_0 = 0$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y''' + 3y'' + 3y' + y = 0$$

Example

Solve the differential equation

$$y''' + 3y'' + 3y' + y = 0$$

Solution

$$y = c_1\, e^x + c_2\, x\, e^x + c_3\, x^2\, e^x$$

Homogeneous LDE with Constant Coefficients

Example

Solve the differential equation

$$y^{(4)} + 2y'' + y = 0$$

Example

Solve the differential equation

$$y^{(4)} + 2y'' + y = 0$$

Solution

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

Homogeneous LDE with Constant Coefficients

Example

Find the differential equation where the roots of the characteristic equation are

$$m = 1, 1, -1, 0, 0$$

Homogeneous LDE with Constant Coefficients

Example

Find the differential equation where the roots of the characteristic equation are

$$m = 1, 1, -1, 0, 0$$

Solution

$$y^{(5)} - y^{(4)} - y''' + y'' = 0$$

Homogeneous LDE with Constant Coefficients

Example

Find the differential equation where the roots of the characteristic equation are

$$m = -1, -1, 0, 3 + 2i, 3 - 2i$$

Example

Find the differential equation where the roots of the characteristic equation are

$$m = -1, -1, 0, 3+2i, 3-2i$$

Solution

$$y^{(5)} - 4y^{(4)} + 2y''' + 20y'' + 13y' = 0$$

Homogeneous LDE with Constant Coefficients

Example

Find a linear differential equation with constant coefficients having solutions

$$1, 5x, 2\cos x, 3\sin x$$

Example

Find a linear differential equation with constant coefficients having solutions

$$1, 5x, 2\cos x, 3\sin x$$

Solution

$$y^{(4)} + y'' = 0$$

Homogeneous LDE with Constant Coefficients

Example

Find a linear differential equation with constant coefficients having solutions

$$2^{3x}, 4\sin 2x + \cos 2x$$

Homogeneous LDE with Constant Coefficients

Example

Find a linear differential equation with constant coefficients having solutions

$$2^{3x}, 4\sin 2x + \cos 2x$$

Solution

$$y''' - 3y'' + 4y' - 12y = 0$$

Nonhomogeneous Linear Differential Equations with Constant Coefficients

Nonhomogeneous Linear Differential Equations

The linear differential equations with constant coefficients has the general form

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = g(x),$$

where $a_n, \ldots, a_1, a_0$ are real constants, and $a_n \neq 0$.

Nonhomogeneous Linear Differential Equations

To find a general solution to this problem

- We first find the general solution y_c to the homogeneous equation

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = 0$$

- We find any particular solution y_p of the nonhomogeneous equation.
- The general solution is

$$y = y_c + y_p$$

Undetermined Coefficient Method

This method is used to find a particular solution y_p to

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = g(x),$$

where $g(x)$ is one of the following functions

- constant k,
- polynomial,
- $e^{\alpha x}$,
- $\sin \beta x$
- $\cos \beta x$
- or finite sums and product of these functions ($x e^{2x} \sin x$)

Undetermined Coefficient Method

Example

$$g(x) = 2$$

$$g(x) = x^2 - x$$

$$g(x) = x - e^{2x}$$

$$g(x) = x^2 \sin 3x$$

Undetermined Coefficient Method

To find the particular solution of

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = g(x),$$

We first find the solution y_c of the homogeneous equation

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0 y = 0,$$

by substituting $y = e^{mx}$, then we obtain the characteristic equation

$$a_n m^n + \ldots + a_1 m + a_0 = 0$$

then we have the following cases:

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = e^{3x}$$

Example

Solve the differential equation

$$y'' - 4y = e^{3x}$$

Solution

$$y = c_1\, e^{2x} + c_2\, e^{-2x} + \frac{e^{3x}}{5}$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = 4x^2 - 8x$$

Example

Solve the differential equation

$$y'' - 4y = 4x^2 - 8x$$

Solution

$$y = c_1\, e^{2x} + c_2\, e^{-2x} - x^2 + 2x - \frac{1}{2}$$

Undetermined Coefficient Method

Case 1: $g(x) = Cx^k e^{rx}$

1. If r is not a root of

$$a_n m^n + \dots + a_1 m + a_0 = 0$$

the the particular solution is

$$y_p = (A_0 + A_1 x + \dots A_k x^k)e^{rx}$$

2. If r is a simple root, then the the particular solution is

$$y_p = x(A_0 + A_1 x + \dots A_k x^k)e^{rx}$$

3. If r is a root of multiplicity q, then the the particular solution is

$$y_p = x^q(A_0 + A_1 x + \dots A_k x^k)e^{rx}$$

Undetermined Coefficient Method

Case 2: $g(x) = Cx^k e^{\alpha x}\cos(\beta x)$ or $g(x) = Cx^k e^{\alpha x}\sin(\beta x)$

1. If $\alpha + i\beta$ is not a root of

$$a_n m^n + ... + a_1 m + a_0 = 0$$

   the the particular solution is

$$y_p = (A_0 + A_1 x + ... A_k x^k)e^{\alpha x}\cos(\beta x) + (B_0 + B_1 x + ... B_k x^k)e^{\alpha x}\sin(\beta x)$$

2. If $\alpha + i\beta$is a simple root, then the the particular solution is

$$y_p = x(A_0 + ... A_k x^k)e^{\alpha x}\cos(\beta x) + x(B_0 + ... B_k x^k)e^{\alpha x}\sin(\beta x)$$

3. If $\alpha + i\beta$ is a root of multiplicity q, then the the particular solution is

$$y_p = x^q(A_0 + ... A_k x^k)e^{\alpha x}\cos(\beta x) + x^q(B_0 + ... B_k x^k)e^{\alpha x}\sin(\beta x)$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = 5xe^{3x}$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = 5xe^{3x}$$

Solution

$$y = c_1\, e^{2x} + c_2\, e^{-2x} + xe^{3x} - \frac{6e^{3x}}{5}$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = e^{2x}$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - 4y = e^{2x}$$

Solution

$$y = c_1\, e^{2x} + c_2\, e^{-2x} + \frac{1}{4} x e^{2x}$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - y = -2x^2 + 5 + 2e^x$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - y = -2x^2 + 5 + 2e^x$$

Solution

$$y = c_1\, e^x + c_2\, e^{-x} + 2x^2 - 1 + xe^x$$

Example

Solve the differential equation

$$y'' - y' = -10\sin 2x$$

Undetermined Coefficient Method

Example

Solve the differential equation

$$y'' - y' = -10\sin 2x$$

Solution

$$y = c_1\, e^{x} + c_2 + 2\sin 2x - \cos 2x$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' - y' = x$$

Example

Determine the form of a particular solution of

$$y'' - y' = x$$

Solution

$$y_p = x(A_0 + A_1 x)$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' - 6y' + 9y = (2x^2 - 1)e^{-x}$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' - 6y' + 9y = (2x^2 - 1)e^{-x}$$

Solution

$$y_p = (A_0 + A_1 x + A_2 x^2)e^{-x}$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' + 4y = x\cos(2x)$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' + 4y = x\cos(2x)$$

Solution

$$y_p = x(A_0 + A_1 x)\cos 2x + x(B_0 + B_1 x)\sin 2x$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' - 6y' + 9y = 6x^2 + 2 - 12x^3e^{3x}$$

Undetermined Coefficient Method

Example

Determine the form of a particular solution of

$$y'' - 6y' + 9y = 6x^2 + 2 - 12x^3e^{3x}$$

Solution

$$y_p = (A_0 + A_1x + A_2x^2) + x^2(B_0 + B_1x + B_2x^2 + B_3x^3)e^{-x}$$