

4 Vector Spaces continued

4.7 Row Space, Column Space, and Null Space

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

Three Fundamental Subspaces

Let $A \in M_{m \times n}$. We have three important vector spaces:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} r_1(A) \\ r_2(A) \\ \vdots \\ r_m(A) \end{bmatrix} = [c_1(A) \quad c_2(A) \quad \cdots \quad c_n(A)]$$

- **Row Space:** $\text{row}(A) = \text{span}\{r_1(A), r_2(A), \dots, r_m(A)\} \subseteq \mathbb{R}^n$
- **Column Space:** $\text{col}(A) = \text{span}\{c_1(A), c_2(A), \dots, c_n(A)\} \subseteq \mathbb{R}^m$
- **Null Space:** $\text{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n$

Example Finding Basis And Dimension of The Fundamental Spaces

Find bases and dimensions for $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix}$.

Solution

$$\text{row}(A) = \text{span}\{ [1 \ 2 \ -1], [3 \ 6 \ -3] \}$$

Note that $r_2(A) = 3 \cdot r_1(A)$, so:

$$\text{row}(A) = \text{span}\{ [1 \ 2 \ -1] \} \subseteq \mathbb{R}^3$$

Basis: $\{ [1 \ 2 \ -1] \}$

Dimension: $\dim(\text{row}(A)) = 1$

Column Space

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$$

Note that $c_2(A) = 2 \cdot c_1(A)$ and $c_3(A) = -1 \cdot c_1(A)$, so:

$$\text{col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

Dimension: $\dim(\text{col}(A)) = 1$

Null Space

To find $\text{null}(A)$, solve $Ax = 0$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{null}(A) = \left\{ \begin{bmatrix} -2t + s \\ t \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Basis: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Dimension: } \dim(\text{null}(A)) = 2$$

Theorem Row Equivalent Matrices Have The Same Row And Null Space.

If $A, B \in M_{m \times n}$ are row equivalent, then:

- $\text{row}(A) = \text{row}(B)$
- $\text{null}(A) = \text{null}(B)$

Proof

Row Space:

- Since rows of B are linear combinations of rows of A : $\text{row}(B) \subseteq \text{row}(A)$
- Since rows of A are linear combinations of rows of B : $\text{row}(A) \subseteq \text{row}(B)$
- Therefore: $\text{row}(A) = \text{row}(B)$

Null Space:

- Elementary row operations do not change the solution set of $Ax = 0$
- Therefore: $\text{null}(A) = \text{null}(B)$

Note Row equivalent matrices may **not** have the same column space.

Theorem: Row Equivalence and Column Spaces

If A and B are row equivalent, then any linear relation between columns of A is satisfied by the corresponding columns of B .

Proof: Let

$$x_1 \mathbf{c}_1(A) + x_2 \mathbf{c}_2(A) + \cdots + x_n \mathbf{c}_n(A) = \mathbf{0} \quad (*)$$

be a linear relation between the columns of A . Since A is row equivalent to B , there exists an invertible matrix F such that $FA = B \implies F\mathbf{c}_i(A) = \mathbf{c}_i(B)$ for all i . Multiplying $(*)$ by F on the left, we get:

$$x_1 \mathbf{c}_1(B) + x_2 \mathbf{c}_2(B) + \cdots + x_n \mathbf{c}_n(B) = \mathbf{0}$$

Which completes the proof.

Theorem: Basis For Row Reduced Echelon Matrices

If R is a row reduced echelon matrix, then:

1. The nonzero rows of R form a basis for $\text{row}(R)$.
2. The pivot columns of R form a basis for $\text{col}(R)$.

Proof:

The results follow from analyzing the positions of the 0's and 1's of R . We omit the details.

Basis For $\text{row}(A)$, $\text{col}(A)$, and $\text{null}(A)$

Let $A \in \mathbb{M}_{m \times n}$. Then:

1. The nonzero rows of $\text{rref}(A)$ ($\text{row ref}(A)$) form a basis for $\text{row}(A)$.
2. The columns of A corresponding to the pivot columns of $\text{rref}(A)$ form a basis for $\text{col}(A)$.
3. The vectors expressing the general solution when solving the system by G.E. or G.J.E.

Example: Finding the solution space (or the null space) of a homogeneous system

Find the null space for matrix:

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Solution: The null space of A is the solution space of $A\mathbf{x} = \mathbf{0}$

$$\text{augmented matrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 \\ 3 & 6 & -5 & 4 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -2s - 3t, x_2 = s, x_3 = -t, x_4 = t$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2$$

$$\Rightarrow \text{null}(A) = \{s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in R\}$$

Example:**Finding a basis for a row space**

Find a basis of the row space of $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \xrightarrow{\text{G. E.}} \text{ref}(A) = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a basis for $\text{row}(A) = \{\text{the nonzero row vectors of } \text{rref}(A)\}$
 $= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

Example: Finding a basis for a subspace using the previous Thm.

Find a basis for the subspace of R^3 spanned by

$$S = \{(-1, 2, 5), (3, 0, 3), (5, 1, 8)\}$$

Solution:

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \begin{matrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{matrix} \xrightarrow{\text{G.E.}} \text{ref}(A) = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{0} \end{matrix}$$

(Construct A such that $\text{row}(A) = \text{span}(S)$)

a basis for $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$

= a basis for $\text{row}(A)$

= {the nonzero row vectors of $\text{ref}(A)$ }

= $\{\mathbf{w}_1, \mathbf{w}_2\}$

= $\{(1, -2, -5), (0, 1, 3)\}$

Example: Finding a basis for the column space of a matrix

Find a basis for the column space of the matrix A given by

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

Solution 1:

Since $col(A) = row(A^T)$, to find a basis for the column space of the matrix A is equivalent to find a basis for the row space of the matrix A^T

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{G. E.}} B = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \end{matrix}$$

a basis for $col(A)$

= a basis for $row(A^T)$

= {the nonzero row vectors of B }

= $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \quad (\text{a basis for the column space of } A)$$

Solution 2:

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{\text{G.J.E.}} B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4$ $\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4$

Leading 1's $\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $col(B)$ (not for $col(A)$)

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $col(A)$

※ This method utilizes that B is with the same dependency relationships among columns as A , which does NOT mean $col(B) = col(A)$

Notes:

The bases for the column space derived from Sol. 1 and Sol. 2 are different. However, both these bases span the same $col(A)$, which is a subspace of R^5

Problem 1

Given a set of vectors $S = \{v_1, v_2, \dots, v_k\}$ in \mathbb{R}^n , find a subset of S that forms a basis for $\text{span}(S)$, and express each of the remaining vectors as a linear combination of the basis vectors.

Solution

Step 1. Form the matrix A whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$

Step 2. Find $\text{rref}(A)$ and denote its column vectors by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$.

Step 3. Identify the columns of $\text{rref}(A)$ that contain the leading 1's. The corresponding column vectors of A form a basis for $\text{span}(S)$. This completes the first part of the problem.

Step 4. Obtain a set of dependency equations for the column vectors $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$ expressing each \mathbf{w}_i that does not contain a leading 1 as a linear combination of predecessors that do.

Step 5. In each dependency equation obtained in Step 4, replace the vector \mathbf{w}_i by the vector \mathbf{v}_i for $i = 1, 2, \dots, k$. This completes the second part of the problem..

Example

(a) Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \mathbf{v}_2 = (2, -5, -3, 6), \mathbf{v}_3 = (0, 1, 3, 0), \mathbf{v}_4 = (2, -1, 4, -7), \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for the subspace of \mathbb{R}^4 spanned by these vectors.

(b) Express each vector *not* in the basis as a linear combination of the basis vectors.

Solution (a) Constructing a Basis

We begin by forming a matrix whose columns are the vectors and reduce to rref:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \longrightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, and 4. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis.

(b) Expressing Other Vectors

We now express the remaining vectors \mathbf{v}_3 and \mathbf{v}_5 as linear combinations of the basis vectors.

From the RREF, we observe:

$$\mathbf{c}_3 = 2\mathbf{c}_1 - \mathbf{c}_2 \quad \Rightarrow \quad \mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

$$\mathbf{c}_5 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4 \quad \Rightarrow \quad \mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$$

These are called the **dependency equations**.

Problem 2

Given a set of L.I. vectors $S = \{v_1, v_2, \dots, v_k\}$ in \mathbb{R}^n , find a basis B of \mathbb{R}^n containing S .

Solution

Step 1. Form the matrix A whose columns are the vectors in the set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$

Step 2. Find $\text{rref}(A)$ and denote its column vectors by $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$.

Step 3. Identify the columns of $\text{rref}(A)$ that contain the leading 1's. The corresponding columns of A (vectors of S) form the desired basis.

Example: Extending a Linearly Independent Set to a Basis

Extend the L.I. set $S = \{\mathbf{v}_1 = (1, 0, 2, 1), \mathbf{v}_2 = (2, 1, 4, 2)\}$ to a basis for \mathbb{R}^4 .

Solution

Step 1: Form the matrix A whose columns are the vectors in S together with the standard basis vectors for \mathbb{R}^4 :

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The columns of matrix A are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$.

Step 2: Compute the reduced row echelon form of A : $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$

Step 3: Identify the columns that contain the leading 1's in $\text{rref}(A)$. These are columns 1, 2, 3, and 5. The corresponding columns of the original matrix A form the desired basis for \mathbb{R}^4 :

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_3\} = \{(1, 0, 2, 1), (2, 1, 4, 2), (1, 0, 0, 0), (0, 0, 1, 0)\}$$

Theorem: $\dim(\text{row}(A)) = \dim(\text{col}(A))$

The row space and the column space of a matrix A have the same dimension.

Proof:

$$\begin{aligned} \dim(\text{row}(A)) &= \dim(\text{row}(\text{rref}(A))) = \text{number of nonzero rows in } \text{rref}(A) \\ &= \text{number of leading 1's (pivot positions)} = \dim(\text{col}(\text{rref}(A))) = \dim(\text{col}(A)) \end{aligned}$$

Definition: Rank and Nullity of a Matrix

- The common dimension of $\text{row}(A)$ and $\text{col}(A)$ is called the **rank** of A and is denoted by $\text{rank}(A)$.
- The dimension of $\text{null}(A)$ is called the **nullity** of A and is denoted by $\text{nullity}(A)$.

Theorem

Dimension Theorem for Matrices

If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These are the leading variables and the free variables. Thus,

$$[\text{number of leading variables}] + [\text{number of free variables}] = n$$

The number of leading variables equals the number of pivots in the row-reduced form of A , which is precisely the rank of A . The number of free variables equals the dimension of the solution space, which is the nullity of A . Therefore $\text{rank}(A) + \text{nullity}(A) = n$.

Example:**Rank and nullity of a matrix**

Let the column vectors of the matrix A be denoted by \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 , \mathbf{a}_4 , and \mathbf{a}_5 .

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}_{4 \times 5}$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5$

- (a) Find the rank and nullity of A
- (b) Find a subset of the column vectors of A that forms a basis for the column space of A
- (c) If possible, write the third column of A as a linear combination of the first two columns

Solution: Derive the reduced row-echelon form of A .

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \Rightarrow rref(A) = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5$ $\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4 \quad \mathbf{b}_5$

(a) $\text{rank}(A) = 3$ (since $\text{rank}(A) =$ the number of nonzero rows in B)

$$\text{nullity}(A) = n - \text{rank}(A) = 5 - 3 = 2 \text{ (by Thm.)}$$

(b) Leading 1's

$\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $\text{col}(rref(A))$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $\text{col}(A)$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \text{ and } \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

(c) $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2 \Rightarrow \mathbf{a}_3 = -2\mathbf{a}_1 + 3\mathbf{a}_2$

The Fundamental Spaces of a Matrix A

- $\text{row}(A)$ $\text{col}(A)$ $\text{null}(A)$
- $\text{row}(A^T)$ $\text{col}(A^T)$ $\text{null}(A^T)$

Theorem: Relations between the fundamental spaces

If $A \in \mathbb{M}_{m \times n}$, then

- $\text{rank}(A) + \text{nullity}(A) = n$
- $\text{rank}(A^T) = \text{rank}(A)$
- $\text{rank}(A^T) + \text{nullity}(A^T) = m$
- $\text{rank}(A) + \text{nullity}(A^T) = m$

Exercise:

True or False?

For any matrix A , $\text{nullity}(A^T) = \text{nullity}(A)$

A) True.

B) False.

 Multiple Choice

Exercise:

True or False?

If A is a square matrix, then $\text{nullity}(A^T) = \text{nullity}(A)$

A) True.

B) False.

 Multiple Choice

Theorem: Solutions of a Nonhomogeneous Linear System

If \mathbf{x}_p is a particular solution to the system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where \mathbf{x}_h is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof: Let \mathbf{x} be another solution of $A\mathbf{x} = \mathbf{b}$ other than \mathbf{x}_p .

$$\Rightarrow A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0} \Rightarrow (\mathbf{x} - \mathbf{x}_p) \text{ is a solution of } A\mathbf{x} = \mathbf{0}.$$

Let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, which is a solution of $A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

Theorem: Relation between the system $Ax=b$ and $\text{col}(A)$

A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A .

Proof: Follows easily from rewriting Ax :

$$Ax = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{c}_1(A) + x_2 \mathbf{c}_2(A) + \cdots + x_n \mathbf{c}_n(A).$$

Example: Finding the solution set of a nonhomogeneous system

Find the set of all solution vectors of the system of linear equations

$$\begin{array}{rcccccc} x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\ 3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\ x_1 & + & 2x_2 & & & - & 5x_4 & = & -9 \end{array}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array} \right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

s *t*

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s & - & t & + & 5 \\ -s & + & 3t & - & 7 \\ s & + & 0t & + & 0 \\ 0s & + & t & + & 0 \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

i.e., $\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution vector of $A\mathbf{x} = \mathbf{b}$,

and $\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$ (you can replace the constant vector with a zero vector to check this result)

Example: Consistency of a system $Ax = b$ depends on whether \mathbf{b} is in $\text{col}(A)$

Check if the following system is consistent by checking for column relations.

$$\begin{aligned}x_1 + x_2 - x_3 &= -1 \\x_1 + x_3 &= 3 \\3x_1 + 2x_2 - x_3 &= 1\end{aligned}$$

Solution:

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & | & -1 \\ 1 & 0 & 1 & | & 3 \\ 3 & 2 & -1 & | & 1 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 0 & 1 & | & 3 \\ 0 & 1 & -2 & | & -4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b} \qquad \mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{v}$

$$\because \mathbf{v} = 3\mathbf{w}_1 - 4\mathbf{w}_2$$

$$\Rightarrow \mathbf{b} = 3\mathbf{a}_1 - 4\mathbf{a}_2 + 0\mathbf{a}_3 \quad (\text{due to the fact that elementary row operations do not change the dependency relationships among columns})$$

(In other words, \mathbf{b} is in the column space of A)

\Rightarrow The system of linear equations is consistent.

- **Note from this example:**

$$\text{rank}(A) = \text{rank}([A | \mathbf{b}]) = 2$$

- **A property that can be inferred:**

If $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$, then the system $A\mathbf{x} = \mathbf{b}$ is consistent

The above property can be analyzed as follows:

- (1) By a Theorem, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} is a linear combination of the columns of A , implies that placing \mathbf{b} to the right of A does NOT increase the number of linearly independent columns, so $\dim(\text{col}(A)) = \dim(\text{col}([A|\mathbf{b}]))$.
- (2) By definition of the rank, $\text{rank}(A) = \dim(\text{col}(A))$ and $\text{rank}([A|\mathbf{b}]) = \dim(\text{col}([A|\mathbf{b}]))$.

Combining (1) and (2), we obtain: $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$ if and only if $A\mathbf{x} = \mathbf{b}$ is consistent.

Theorem: Summary of equivalent conditions for square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent

- (1) A is invertible
- (2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix \mathbf{b}
- (3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
- (4) A is row-equivalent to I_n
- (5) $\det(A) \neq 0$
- (6) $\text{rank}(A) = n$
- (7) There are n row vectors of A which are linearly independent
- (8) There are n column vectors of A which are linearly independent