4 Vector Spaces

4.7 Row Space, Column Space, and Null Space

4.8 Rank, Nullity, and the Fundamental Matrix Spaces

Three Fundamental Subspaces

Let $A \in M_{m\times n}$. We have three important vector spaces:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} r_1(A) \\ r_2(A) \\ \vdots \\ r_m(A) \end{bmatrix} = \begin{bmatrix} c_1(A) & c_2(A) & \cdots & c_n(A) \end{bmatrix}$$

- **Row Space**: $\mathrm{row}(A) = \mathrm{span}\{\mathrm{r}_1(A), \mathrm{r}_2(A), \ldots, \mathrm{r}_m(A)\} \subseteq \mathbb{R}^n$
- **Column Space**: $\mathrm{col}(A) = \mathrm{span}\{\mathrm{c}_1(A), \mathrm{c}_2(A), \ldots, \mathrm{c}_n(A)\} \subseteq \mathbb{R}^m$
- **Null Space**: $\mathrm{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n$

Example Finding Basis And Dimension of The Fundamental Spaces

Find bases and dimensions for $\mathrm{row}(A)$, $\mathrm{col}(A)$, and $\mathrm{null}(A)$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix}$.

Solution

$$\mathrm{row}(A) = \mathrm{span}\{\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 6 & -3 \end{bmatrix}\}$$

Note that $\mathrm{r}_2(A) = 3 \cdot \mathrm{r}_1(A)$, so:

$$\mathrm{row}(A) = \mathrm{span}\{\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\} \subseteq \mathbb{R}^3$$

Basis: $\{\begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\}$

Dimension: $\dim(\mathrm{row}(A)) = 1$

Column Space

$$\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\}$$

Note that $\mathrm{c}_2(A) = 2 \cdot \mathrm{c}_1(A)$ and $\mathrm{c}_3(A) = -1 \cdot \mathrm{c}_1(A)$, so:

$$\mathrm{col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

Dimension: $\dim(\mathrm{col}(A)) = 1$

Null Space

To find $\text{null}(A)$, solve $Ax = 0$:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{null}(A) = \left\{ \begin{bmatrix} -2t + s \\ t \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Dimension: $\dim(\text{null}(A)) = 2$

Theorem Row Equivalent Matrices Have The Same Row And Null Space.

If $A, B \in M_{m\times n}$ are row equivalent, then:

- $\text{row}(A) = \text{row}(B)$
- $\text{null}(A) = \text{null}(B)$

Proof

Row Space:

- Since rows of B are linear combinations of rows of A: $\text{row}(B) \subseteq \text{row}(A)$
- Since rows of A are linear combinations of rows of B: $\text{row}(A) \subseteq \text{row}(B)$
- Therefore: $\text{row}(A) = \text{row}(B)$

Null Space:

- Elementary row operations do not change the solution set of $Ax = 0$
- Therefore: $\text{null}(A) = \text{null}(B)$

Note Row equivalent matrices may **not** have the same column space.

Theorem: Row Equivalence and Column Spaces

If A and B are row equivalent, then any linear relation between columns of A is satisfied by the corresponding columns of B.

Proof: Let

$$x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + \cdots + x_n\mathbf{c}_n(A) = \mathbf{0} \quad (*)$$

be a linear relation between the columns of A. Since A is row equivalent to B, there exists an invertible matrix F such that $FA = B \implies F\mathbf{c}_i(A) = \mathbf{c}_i(B)$ for all i. Multiplying $(*)$ by F on the left, we get:

$$x_1\mathbf{c}_1(B) + x_2\mathbf{c}_2(B) + \cdots + x_n\mathbf{c}_n(B) = \mathbf{0}$$

Which completes the proof.

Theorem: Basis For Row Reduced Echelon Matrices

If R is a row reduced echelon matrix, then:

1. The nonzero rows of R form a basis for $\mathrm{row}(R)$.
2. The pivot columns of R form a basis for $\mathrm{col}(R)$.

Proof:

The results follow from analyzing the positions of the 0's and 1's of R. We omit the details.

Basis For row (A), col(A), and null(A)

Let $A \in \mathbb{M}_{m \times n}$. Then:

1. The nonzero rows of $\mathrm{rref}(A)$ (ro $\mathrm{ref}(A)$) form a basis for $\mathrm{row}(A)$.
2. The columns of A corresponding to the pivot columns of $\mathrm{rref}(A)$ form a basis for $\mathrm{col}(A)$.
3. The vectors expressing the general solution when solving the system by G.E. or G.J.E.

Example: Finding the solution space (or the null space) of a homogeneous system

Find the null space for matrix:

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

Solution: The null space of A is the solution space of $A\mathbf{x} = \mathbf{0}$

$$\text{augmented matrix} = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 \\ 3 & 6 & -5 & 4 & 0 \\ 1 & 2 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{\text{G.-J. E.}} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_1 = -2s - 3t,\ x_2 = s,\ x_3 = -t,\ x_4 = t$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2$$

$$\Rightarrow null(A) = \{s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in R\}$$

Example: Finding a basis for a row space

Find a basis of the row space of $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \xrightarrow{\text{G. E.}} ref(A) = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a basis for $row(A)$ = {the nonzero row vectors of rref(A)}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = \{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

Example: Finding a basis for a subspace using the previous Thm.

Find a basis for the subspace of R^3 spanned by

$$S = \{\overset{\mathbf{v}_1}{(-1, 2, 5)}, \overset{\mathbf{v}_2}{(3, 0, 3)}, \overset{\mathbf{v}_3}{(5, 1, 8)}\}$$

Solution:

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \xrightarrow{\text{G. E.}} \operatorname{ref}(A) = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \end{array}$$

(Construct A such that $row(A)$ = span(S))

a basis for span($\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$)

= a basis for $row(A)$

= {the nonzero row vectors of ref(A)}

$= \{\mathbf{w}_1, \mathbf{w}_2\}$

$= \{(1, -2, -5), (0, 1, 3)\}$

Example: Finding a basis for the column space of a matrix

Find a basis for the column space of the matrix A given by

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

Solution 1:

Since $col(A)=row(A^T)$, to find a basis for the column space of the matrix A is equivalent to find a basis for the row space of the matrix A^T

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{G. E.}} B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \end{array}$$

$$\begin{aligned}
&\text{a basis for } col(A)\\
&= \text{ a basis for } row(A^T)\\
&= \{\text{the nonzero row vectors of } B\}\\
&= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}
\end{aligned}$$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \quad \text{(a basis for the column space of } A\text{)}$$

Solution 2:

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{\text{G.J. E.}} B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \qquad\qquad\qquad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4$$

Leading 1's $\Rightarrow$ $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $col(B)$ (not for $col(A)$)

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $col(A)$

※ This method utilizes that B is with the same dependency relationships among columns as A, which does NOT mean $col(B) = col(A)$

Notes:

The bases for the column space derived from Sol. 1 and Sol. 2 are different. However, both these bases span the same $col(A)$, which is a subspace of R^5

Problem 1

Given a set of vectors $S = \{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$, find a subset of S that forms a basis for span(S), and express each of the remining vectors as a linear combination of the basis vectors.

Solution

Step 1. Form the matrix A whose columns are the vectors in the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$

Step 2. Find rref(A) and denote its column vectors by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

Step 3. Identify the columns of rref(A) that contain the leading 1's. The corresponding column vectors of A form a basis for $\operatorname{span}(S)$. This completes the first part of the problem.

Step 4. Obtain a set of dependency equations for the column vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ expressing each $\mathbf{w}_i$ that does not contain a leading 1 as a linear combination of predecessors that do.

Step 5. In each dependency equation obtained in Step 4, replace the vector $\mathbf{w}_i$ by the vector $\mathbf{v}_i$ for $i = 1, 2, \ldots, k$. This completes the second part of the problem..

Example

(a) Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3),\ \mathbf{v}_2 = (2, -5, -3, 6),\ \mathbf{v}_3 = (0, 1, 3, 0),\ \mathbf{v}_4 = (2, -1, 4, -7),\ \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for the subspace of $\mathbb{R}^4$ spanned by these vectors.

(b) Express each vector *not* in the basis as a linear combination of the basis vectors.

Solution (a) Constructing a Basis

We begin by forming a matrix whose columns are the vectors and reduce to rref:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \longrightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 2, and 4. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis.

(b) Expressing Other Vectors

We now express the remaining vectors $\mathbf{v}_3$ and $\mathbf{v}_5$ as linear combinations of the basis vectors.

From the RREF, we observe:

$$\mathbf{c}_3 = 2\mathbf{c}_1 - \mathbf{c}_2 \quad \Rightarrow \quad \mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$

$$\mathbf{c}_5 = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_4 \quad \Rightarrow \quad \mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$$

These are called the **dependency equations.**

Problem 2

Given a set of L.I. vectors $S = \{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$, find a basis B of $\mathbb{R}^n$ containing S.

Solution

Step 1. Form the matrix A whose columns are the vectors in the set
$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k,\ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

Step 2. Find $\mathrm{rref}(A)$ and denote its column vectors by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

Step 3. Identify the columns of $\mathrm{rref}(A)$ that contain the leading 1's. The corresponding columns of A(vectors of S) form the desired basis.

Example: Extending a Linearly Independent Set to a Basis

Extend the L.I. set $S = \{\mathbf{v}_1 = (1,0,2,1), \mathbf{v}_2 = (2,1,4,2)\}$ to a basis for $\mathbb{R}^4$.

Solution

Step 1: Form the matrix A whose columns are the vectors in S together with the standard basis vectors for $\mathbb{R}^4$:

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The columns of matrix A are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$.

Step 2: Compute the reduced row echelon form of A: $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$

Step 3: Identify the columns that contain the leading 1's in $\mathrm{rref}(A)$. These are columns 1, 2, 3, and 5. The corresponding columns of the original matrix A form the desired basis for $\mathbb{R}^4$:

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_3\} = \{(1,0,2,1), (2,1,4,2), (1,0,0,0), (0,0,1,0)\}$$

Theorem: $\dim(\text{row}(A)) = \dim(\text{col}(A))$

The row space and the column space of a matrix A have the same dimension.

Proof:

$$\dim(\text{row}(A)) = \dim(\text{row}(\text{rref}(A))) = \text{number of nonzero rows in rref}(A)$$
$$= \text{number of leading 1's (pivot positions)} = \dim(\text{col}(\text{rref}(A))) = \dim(\text{col}(A))$$

Definition: Rank and Nullity of a Matrix

- The common dimension of $\text{row}(A)$ and $\text{col}(A)$ is called the **rank** of A and is denoted by $\text{rank}(A)$.
- The dimension of $\text{null}(A)$ is called the **nullity** of A and is denoted by $\text{nullity}(A)$.

Theorem Dimension Theorem for Matrices

If A is a matrix with n columns, then $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$

Proof Since A has n columns, the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has n unknowns (variables). These are the leading variables and the free variables. Thus,

$$[\text{number of leading variables}] + [\text{number of free variables}] = n$$

The number of leading variables equals the number of pivots in the row-reduced form of A, which is precisely the rank of A. The number of free variables equals the dimension of the solution space, which is the nullity of A. Therefore $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

Example: Rank and nullity of a matrix

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix}_{4\times 5} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of A

(b) Find a subset of the column vectors of A that forms a basis for the column space of A

(c) If possible, write the third column of A as a linear combination of the first two columns

Solution: Derive the reduced row-echelon form of A.

$$A=\begin{bmatrix}1&0&-2&1&0\\0&-1&-3&1&3\\-2&-1&1&-1&3\\0&3&9&0&-12\end{bmatrix}\Rightarrow rref(A)=\begin{bmatrix}1&0&-2&0&1\\0&1&3&0&-4\\0&0&0&1&-1\\0&0&0&0&0\end{bmatrix}$$

$$\begin{array}{ccccc}\mathbf{a}_1&\mathbf{a}_2&\mathbf{a}_3&\mathbf{a}_4&\mathbf{a}_5\end{array}\qquad\qquad\begin{array}{ccccc}\mathbf{b}_1&\mathbf{b}_2&\mathbf{b}_3&\mathbf{b}_4&\mathbf{b}_5\end{array}$$

(a) $\operatorname{rank}(A)=3$ (since rank(A) = the number of nonzero rows in B)

$\text{nuillity}(A)=n-\operatorname{rank}(A)=5-3=2$ (by Thm.)

(b) Leading 1's

$\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for col($rref\,(A)$)

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for col(A)

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \ \text{and} \ \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

(c) $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2 \ \Rightarrow \ \mathbf{a}_3 = -2\mathbf{a}_1 + 3\mathbf{a}_2$

The Fundamental Spaces of a Matrix A

- $\text{row}(A)$ $\quad\quad$ $\text{col}(A)$ $\quad\quad$ $\text{null}(A)$
- $\text{row}(A^T)$ $\quad\quad$ $\text{col}(A^T)$ $\quad\quad$ $\text{null}(A^T)$

Theorem: Relations between the fundamental spaces

If $A \in \mathbb{M}_{m\times n}$, then

- $\text{rank}(A) + \text{nullity}(A) = n$
- $\text{rank}(A^T) = \text{rank}(A)$
- $\text{rank}(A^T) + \text{nullity}(A^T) = m$
- $\text{rank}(A) + \text{nullity}(A^T) = m$

Theorem: Solutions of a Nonhomogeneous Linear System

If $\mathbf{x}_p$ is a particular solution to the system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof: Let $\mathbf{x}$ be another solution of $A\mathbf{x} = \mathbf{b}$ other than $\mathbf{x}_p$.

$\Rightarrow A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0} \Rightarrow (\mathbf{x} - \mathbf{x}_p)$ is a solution of $A\mathbf{x} = \mathbf{0}$.

Let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$, which is a solution of $A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

Theorem: Relation between the system Ax=b and col(A)

A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A.

Proof: Follows easily from rewriting Ax:

$$Ax = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{c}_1(A) + x_2 \mathbf{c}_2(A) + \cdots + x_n \mathbf{c}_n(A) \,.$$

Example: Finding the solution set of a nonhomogeneous system

Find the set of all solution vectors of the system of linear equations

$$\begin{array}{rcrcrcrcr}
x_1 & & & - & 2x_3 & + & x_4 & = & 5 \\
3x_1 & + & x_2 & - & 5x_3 & & & = & 8 \\
x_1 & + & 2x_2 & & & - & 5x_4 & = & -9
\end{array}$$

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \begin{array}{c} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{rrrrr} & & s & t & \end{array} \end{array}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s & - & t & + & 5 \\ -s & + & 3t & - & 7 \\ s & + & 0t & + & 0 \\ 0s & + & t & + & 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

i.e., $\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution vector of $A\mathbf{x} = \mathbf{b}$,

and $\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$ (you can replace the constant vector with a zero vector to check this result)

Example: Consistency of a system $Ax = b$ depends on whether **b** is in col(A)

Check if the following system is consistent by checking for column relations.

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -1 \\ x_1 & & & + & x_3 & = & 3 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 1 \end{array}$$

Solution:

$$[A \,\vdots\, \mathbf{b}] = \left[\begin{array}{rrr:r} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{rrr:r} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array} \qquad\qquad \begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{v} \end{array}$$

$$\because \mathbf{v} = 3\mathbf{w}_1 - 4\mathbf{w}_2$$

$$\Rightarrow \mathbf{b} = 3\mathbf{a}_1 - 4\mathbf{a}_2 + 0\mathbf{a}_3$$ (due to the fact that elementary row operations do not change the dependency relationships among columns)

(In other words, **b** is in the column space of A)

$\Rightarrow$ The system of linear equations is consistent.

- Note from this example:

$$\operatorname{rank}(A) = \operatorname{rank}([A \mid \mathbf{b}]) = 2$$

- A property that can be inferred:

If $\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}])$, then the system $A\mathbf{x} = \mathbf{b}$ is consistent

The above property can be analyzed as follows:

(1) By a Theorem, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A, implies that placing $\mathbf{b}$ to the right of A does NOT increase the number of linearly independent columns, so $\dim(col(A)) = \dim(col([A|\mathbf{b}]))$.

(2) By definition of the rank, $\operatorname{rank}(A) = \dim(col(A))$ and $\operatorname{rank}([A|\mathbf{b}]) = \dim(col([A|\mathbf{b}]))$.

Combining (1) and (2), we obtain: $\operatorname{rank}(A) = \operatorname{rank}([A|\mathbf{b}])$ if and only if $A\mathbf{x} = \mathbf{b}$ is consistent.

Theorem: Summary of equivalent conditions for square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $\det(A) \neq 0$

(6) $\text{rank}(A) = n$

(7) There are n row vectors of A which are linearly independent

(8) There are n column vectors of A which are linearly independent