UNIT 3C

The Fundamental Matrix Spaces

Unit 3C · Sections 4.7 – 4.8

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

What you should be able to do

Section 4.7

- Describe the row space, the column space and the null space of a matrix, and say in which $\mathbb{R}^k$ each one lives.
- Decide from the column space whether $A\mathbf{x} = \mathbf{b}$ is consistent, and write every solution as one particular solution plus a solution of $A\mathbf{x} = \mathbf{0}$.
- Find bases for the three spaces from a row echelon form, find a basis for a span inside the given set together with the dependency equations, and extend an independent set to a basis of $\mathbb{R}^n$.

Section 4.8

- Find the rank and the nullity of a matrix, and use $\text{rank}(A) + \text{nullity}(A) = n$.
- Give the dimensions of the four fundamental spaces, decide consistency from ranks, and use the Equivalence Theorem for square matrices.

SECTION 4.7

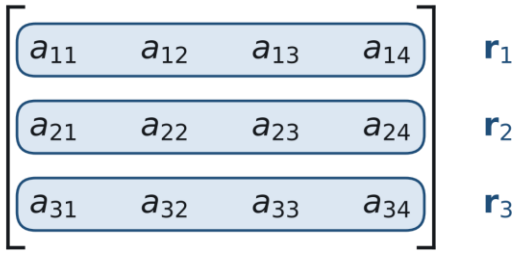
Row Space, Column Space, and Null Space

GOAL Every matrix comes with three subspaces: its row space, its column space and its null space. They tell us when $A\mathbf{x} = \mathbf{b}$ has a solution and what all of its solutions look like, and row reduction gives a basis for each of them.

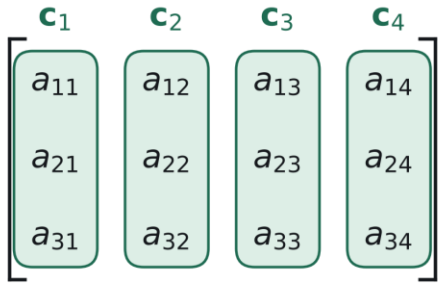
1. Row vectors and column vectors
2. Row space, column space and null space
3. Consistency and the column space
4. All solutions: particular + homogeneous
5. Bases from a row echelon form
6. A basis for a span, and dependency equations
7. Extending an independent set to a basis

Row vectors and column vectors

DEFINITION 1 Let A be an $m\times n$ matrix. Its m rows $\mathbf{r}_1, \dots, \mathbf{r}_m$, written as $1\times n$ matrices, are the ROW VECTORS of A: vectors in $\mathbb{R}^n$. Its n columns $\mathbf{c}_1, \dots, \mathbf{c}_n$, written as $m\times 1$ matrices, are the COLUMN VECTORS of A: vectors in $\mathbb{R}^m$.



rows: three vectors in $\mathbb{R}^4$



columns: four vectors in $\mathbb{R}^3$

Example 1

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}: \qquad \mathbf{r}_1 = [2 \quad 1 \quad 0],\ \mathbf{r}_2 = [3 \quad -1 \quad 4] \text{ in } \mathbb{R}^3; \qquad \mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \text{ in } \mathbb{R}^2$$

DEFINITION 2

The row space, the column space and the null space

DEFINITION 2 Let A be an $m \times n$ matrix.

1) The ROW SPACE of A is $\text{row}(A) = \text{span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}$, a subspace of $\mathbb{R}^n$.

2) The COLUMN SPACE of A is $\text{col}(A) = \text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$, a subspace of $\mathbb{R}^m$.

3) The NULL SPACE of A is $\text{null}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$, the solution space of $A\mathbf{x} = \mathbf{0}$, a subspace of $\mathbb{R}^n$.

	$\text{row}(A)$	$\text{col}(A)$	$\text{null}(A)$
Made from	the rows of A	the columns of A	the solutions of $A\mathbf{x} = \mathbf{0}$
Lives in	$\mathbb{R}^n$	$\mathbb{R}^m$	$\mathbb{R}^n$

Two questions for Sections 4.7 and 4.8:

- How are the solutions of $A\mathbf{x} = \mathbf{b}$ related to these three spaces?
- How are the three spaces related to each other?

EXAMPLE

The three spaces of a 2×3 matrix

Find a basis and the dimension of $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix}$.

Row space

$\mathbf{r}_2 = 3\mathbf{r}_1$, so $\text{row}(A) = \text{span}\{\mathbf{r}_1\}$ with basis $\{[1 \quad 2 \quad -1]\}$ and dimension 1.

Column space

$\mathbf{c}_2 = 2\mathbf{c}_1$ and $\mathbf{c}_3 = -\mathbf{c}_1$, so $\text{col}(A) = \text{span}\{\mathbf{c}_1\}$ with basis $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ and dimension 1.

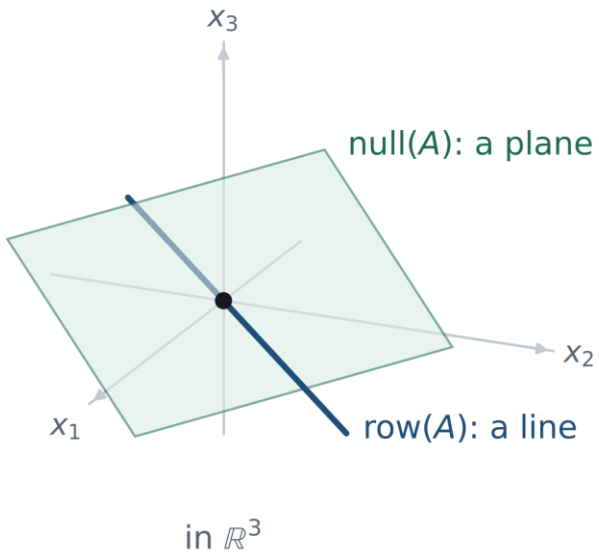
Null space

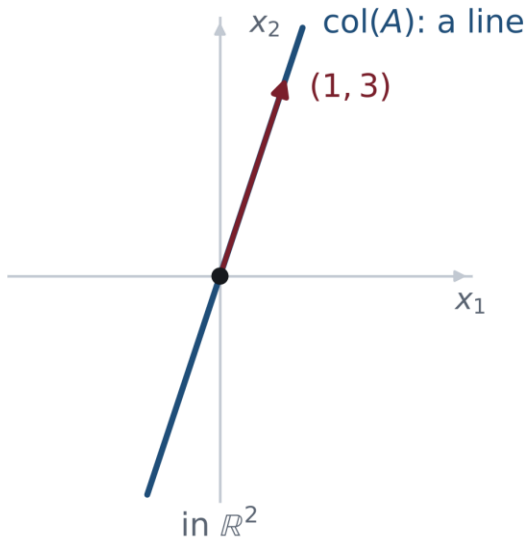
$A\mathbf{x} = \mathbf{0}$ reduces to the single equation $x_1 + 2x_2 - x_3 = 0$. With $x_2 = s$ and $x_3 = t$ free, $x_1 = -2s + t$:

$$\text{null}(A) = \left\{ s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Basis $\{(-2,1,0),\ (1,0,1)\}$, dimension 2.

Where the three spaces live





- $\text{row}(A)$ and $\text{null}(A)$ lie in $\mathbb{R}^3$ because A has 3 columns: $\text{row}(A)$ is a line (dimension 1) and $\text{null}(A)$ is a plane (dimension 2). $\text{col}(A)$ lies in $\mathbb{R}^2$ because A has 2 rows; it is a line.

$\dim\text{row}(A) + \dim\text{null}(A) = 1 + 2 = 3$, the number of columns of A. Section 4.8 shows that this always happens.

When does $A\mathbf{x} = \mathbf{b}$ have a solution?

If $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are the columns of A, then $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$ (Section 1.3). So solving $A\mathbf{x} = \mathbf{b}$ means writing $\mathbf{b}$ as a combination of the columns.

> **THEOREM 4.7.1** A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{col}(A)$.

Example 2: show that $\mathbf{b} = (1, -9, -3) \in \text{col}(A)$

$$\left[\begin{array}{cccc} -1 & 3 & 2 & 1 \\ 1 & 2 & -3 & -9 \\ 2 & 1 & -2 & -3 \end{array}\right] \xrightarrow{I_{1,2},\, A_{1,2}^{(1)},\, A_{1,3}^{(-2)}} \left[\begin{array}{cccc} 1 & 2 & -3 & -9 \\ 0 & 5 & -1 & -8 \\ 0 & -3 & 4 & 15 \end{array}\right] \xrightarrow{A_{3,2}^{(2)},\, M_2^{(-1)},\, A_{2,3}^{(3)}} \left[\begin{array}{cccc} 1 & 2 & -3 & -9 \\ 0 & 1 & -7 & -22 \\ 0 & 0 & -17 & -51 \end{array}\right]$$

Back-substitution: $x_3 = 3,\ x_2 = -22 + 7x_3 = -1,\ x_1 = -9 - 2x_2 + 3x_3 = 2$. The system is consistent, and

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

THEOREM 4.7.2

All solutions of $A\mathbf{x} = \mathbf{b}$

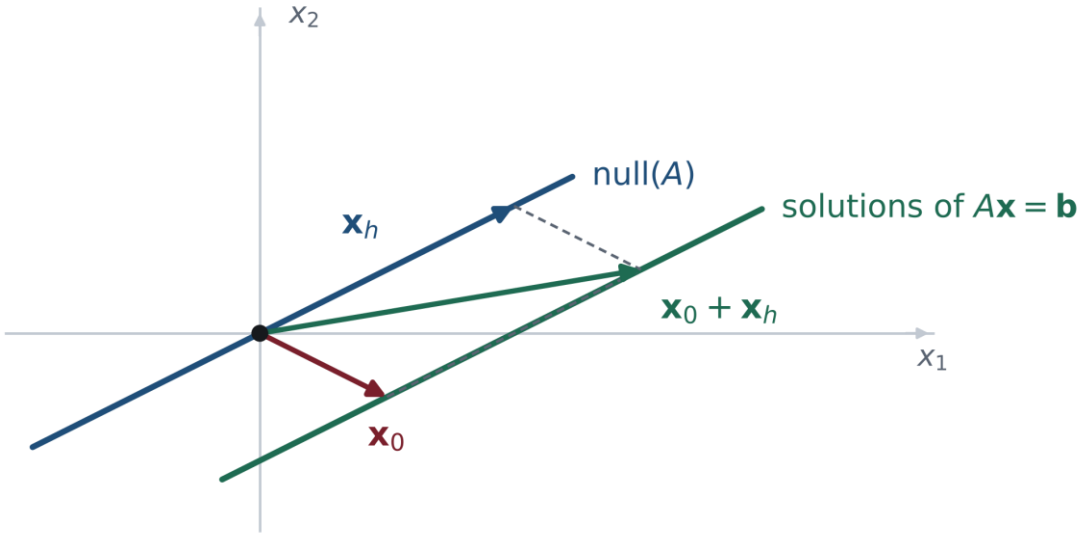
THEOREM 4.7.2 Let $\mathbf{x}_0$ be one solution of a consistent system $A\mathbf{x} = \mathbf{b}$, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a basis for $\text{null}(A)$. Then the solutions of $A\mathbf{x} = \mathbf{b}$ are exactly the vectors $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ with $c_1, \dots, c_k \in \mathbb{R}$.

Why

If $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{x} - \mathbf{x}_0) = \mathbf{b} - \mathbf{b} = \mathbf{0}$, so $\mathbf{x} - \mathbf{x}_0 \in \text{null}(A)$ is a combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Conversely, if $\mathbf{x}_h \in \text{null}(A)$, then $A(\mathbf{x}_0 + \mathbf{x}_h) = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

$\mathbf{x}_0$ is a PARTICULAR solution; $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is the general solution of $A\mathbf{x} = \mathbf{0}$.



General solution of $A\mathbf{x} = \mathbf{b}$ = one particular solution + the general solution of $A\mathbf{x} = \mathbf{0}$. The solution set is $\text{null}(A)$ moved by $\mathbf{x}_0$.

EXAMPLE

Solving a nonhomogeneous system

Find all solutions of

$$\begin{aligned} x_1 - 2x_3 + x_4 &= 5 \\ 3x_1 + x_2 - 5x_3 &= 8 \\ x_1 + 2x_2 - 5x_4 &= -9 \end{aligned}$$

Plan: row reduce $[\,A \mid \mathbf{b}\,]$, solve for the leading variables, and split the answer as in Theorem 4.7.2.

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \overset{A_{1,2}^{(-3)},\, A_{1,3}^{(-1)}}{\rightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 2 & 2 & -6 & -14 \end{array}\right] \overset{A_{2,3}^{(-2)}}{\rightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ and $x_4 = t$ are free; $x_1 = 5 + 2s - t$, $x_2 = -7 - s + 3t$. In vector form:

$$\mathbf{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \mathbf{x}_0 + s\mathbf{u}_1 + t\mathbf{u}_2$$

- $\mathbf{x}_0 = (5, -7, 0, 0)$ is a particular solution (take $s = t = 0$), and $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\text{null}(A)$: with right side $\mathbf{0}$, the same reduction gives $\mathbf{x} = s\mathbf{u}_1 + t\mathbf{u}_2$.

EXAMPLES 3 AND 4

The 4×6 systems of Section 1.2 again

$$A=\begin{bmatrix}1&3&-2&0&2&0\\2&6&-5&-2&4&-3\\0&0&5&10&0&15\\2&6&0&8&4&18\end{bmatrix},\qquad \mathbf{b}=\begin{bmatrix}0\\-1\\5\\6\end{bmatrix}$$

Example 4: a basis for $\text{null}(A)$. The system $A\mathbf{x}=\mathbf{0}$ is Example 6 of Section 1.2, solved again in Section 4.5 (Example 3); its general solution gives

$$\mathbf{v}_1=(-3,1,0,0,0,0),\quad \mathbf{v}_2=(-4,0,-2,1,0,0),\quad \mathbf{v}_3=(-2,0,0,0,1,0)$$

Example 3: $A\mathbf{x}=\mathbf{b}$ is Example 5 of Section 1.2. Its general solution with all parameters 0 is $\mathbf{x}_0=\left(0,0,0,0,0,\frac{1}{3}\right)$; check: $A\mathbf{x}_0$ is $\frac{1}{3}$ times the last column $(0,-3,15,18)$ of A, which is $\mathbf{b}$. By Theorem 4.7.2 every solution is

$$\mathbf{x}=\mathbf{x}_0+r\mathbf{v}_1+s\mathbf{v}_2+t\mathbf{v}_3,\qquad r,s,t\in\mathbb{R}$$

Each $\mathbf{v}_i$ comes from setting one parameter equal to 1 and the others equal to 0 (Anton's Remark).

What row operations keep, and what they change

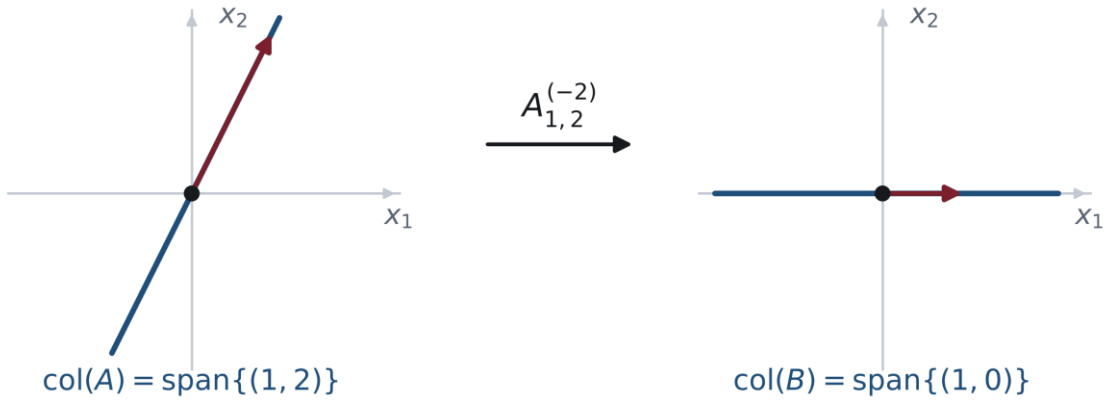
THEOREMS 4.7.3 AND 4.7.4 Elementary row operations do not change the null space or the row space of a matrix: if A and B are row equivalent, then $\text{null}(A) = \text{null}(B)$ and $\text{row}(A) = \text{row}(B)$.

Why

- Null space: row operations do not change the solutions of $A\mathbf{x} = \mathbf{0}$.
- Row space: each row of B is a combination of rows of A, so $\text{row}(B) \subseteq \text{row}(A)$; undoing the operations gives $\text{row}(A) \subseteq \text{row}(B)$.

The column space CAN change

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \overset{A_{1,2}^{(-2)}}{\rightarrow} B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$



Bases for a matrix in row echelon form

THEOREM 4.7.5 Let R be a matrix in row echelon form.

1) The nonzero rows of R (the rows with leading 1's) form a basis for $\text{row}(R)$.

2) The columns of R that contain the leading 1's form a basis for $\text{col}(R)$.

Example 5

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for $\text{row}(R)$: the three nonzero rows

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -2 \quad 5 \quad 0 \quad 3], \\ \mathbf{r}_2 &= [0 \quad 1 \quad 3 \quad 0 \quad 0], \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

Basis for $\text{col}(R)$: columns 1, 2 and 4

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE 6

A basis for the row space by row reduction

Row operations do not change the row space (Theorem 4.7.4): reduce A to a row echelon form R and take the nonzero rows of R.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \xrightarrow{A_{1,2}^{(-2)},\, A_{1,3}^{(-2)},\, A_{1,4}^{(1)}} \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{A_{2,3}^{(-1)}} \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The nonzero rows of R form a basis for $\mathrm{row}(R) = \mathrm{row}(A)$:

$$\mathbf{r}_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4], \quad \mathbf{r}_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6], \quad \mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

Only $\mathbf{r}_1$ is a row of A. Example 9 shows, for another matrix, how to get a basis made of rows of the matrix itself.

THEOREM 4.7.6

Row operations keep the relations among the columns

THEOREM 4.7.6 Let A and B be row equivalent matrices.

(a) A set of columns of A is linearly independent if and only if the corresponding columns of B are linearly independent.

(b) A set of columns of A is a basis for $\mathrm{col}(A)$ if and only if the corresponding columns of B form a basis for $\mathrm{col}(B)$.

Why

A relation among the columns of A with coefficients $x_1, \dots, x_n$,

$$x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + \cdots + x_n\mathbf{c}_n(A) = \mathbf{0},$$

says exactly that $\mathbf{x} = (x_1, \dots, x_n)$ solves $A\mathbf{x} = \mathbf{0}$. Since $\mathrm{null}(A) = \mathrm{null}(B)$, the same $\mathbf{x}$ solves $B\mathbf{x} = \mathbf{0}$: the columns of B satisfy the SAME relation, with the same coefficients.

Both parts are statements about such relations: a set of columns is linearly independent when its only relation is the trivial one, and it is a basis for the column space when, in addition, every other column is a combination of it.

EXAMPLE 7

A basis for the column space, made of columns of A

A from Example 6 and its row echelon form R: the leading 1's of R are in columns 1, 3 and 5.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1, 3, 5 of R form a basis for $\mathrm{col}(R)$ (Theorem 4.7.5), so by Theorem 4.7.6(b) columns 1, 3, 5 of A form a basis for $\mathrm{col}(A)$:

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \qquad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Columns 1, 3, 5 of R are NOT a basis for $\mathrm{col}(A)$: every combination of columns of R ends in 0, but $\mathbf{c}_1 = (1,2,2,-1) \in \mathrm{col}(A)$ does not.

Example 8: if $\mathbf{v}_1, \ldots, \mathbf{v}_6$ are the columns of A, then $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ is a basis for $\mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_6\} = \mathrm{col}(A)$.

METHOD

Bases for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$

METHOD Reduce A to its reduced row echelon form R.

1) $\text{row}(A)$: the nonzero rows of R.

2) $\text{col}(A)$: the columns of A in the positions of the leading 1's of R (not the columns of R).

3) $\text{null}(A)$: solve $R\mathbf{x} = \mathbf{0}$; take the vectors that multiply the free parameters.

Example

$$A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{bmatrix} \xrightarrow{A_{1,2}^{(-3)},\, A_{1,3}^{(-1)}} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \xrightarrow{A_{2,3}^{(-2)},\, A_{2,1}^{(2)}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\text{row}(A)$: basis $\{(1,2,0,3),\ (0,0,1,1)\}$.

$\text{col}(A)$: leading 1's in columns 1 and 3, so the basis is $\{(1,3,1),\ (-2,-5,0)\}$, columns 1 and 3 of A.

$\text{null}(A)$: $x_2 = s$ and $x_4 = t$ are free; $x_1 = -2s - 3t$ and $x_3 = -t$, so

$$\mathbf{x} = s(-2,1,0,0) + t(-3,0,-1,1)$$

and the basis is $\{(-2,1,0,0),\ (-3,0,-1,1)\}$.

EXAMPLE

A basis for a span: put the vectors in the rows

Find a basis for the subspace of $\mathbb{R}^3$ spanned by $S = \{(-1,2,5),\ (3,0,3),\ (5,1,8)\}$.

The span of S is the row space of the matrix with these vectors as ROWS. Row reduce it:

$$\begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \overset{M_2^{(1/3)},\ I_{1,2}}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 5 \\ 5 & 1 & 8 \end{bmatrix} \overset{A_{1,2}^{(1)},\ A_{1,3}^{(-5)}}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} \overset{I_{2,3},\ A_{2,3}^{(-2)}}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows give the basis $\{(1,0,1),\ (0,1,3)\}$ for $\text{span}(S)$, which has dimension 2.

> These basis vectors are simple, but they are not vectors of S. To find a basis made of vectors of S, put the vectors in the COLUMNS instead (Example 10).

EXAMPLE 9

A basis for the row space, made of rows of A

Transpose: $\text{row}(A) = \text{col}(A^T)$. Find a basis for $\text{col}(A^T)$ made of columns of A^T (as in Example 7), then transpose back.

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

$$A^T \xrightarrow{A_{1,2}^{(2)},\, A_{1,5}^{(-3)}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -1 & 5 & 10 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{M_2^{(-1)},\, A_{2,3}^{(3)},\, A_{2,4}^{(2)}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{M_3^{(-1/12)},\, A_{3,4}^{(12)}} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1's in columns 1, 2 and 4, so rows 1, 2 and 4 of A form a basis for $\text{row}(A)$:

$$\mathbf{r}_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3], \quad \mathbf{r}_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6], \quad \mathbf{r}_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]$$

PROCEDURE

A basis taken from S, and the dependency equations

PROCEDURE Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be vectors in $\mathbb{R}^n$.

1) Form the matrix A whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
2) Find its reduced row echelon form R, and call the columns of R $\mathbf{w}_1, \ldots, \mathbf{w}_k$.
3) The leading 1's of R mark the vectors of S that form a basis for $\text{span}(S)$.
4) Write each other $\mathbf{w}_i$ as a combination of the leading-1 columns before it; read the coefficients off R.
5) Replace every $\mathbf{w}$ by the matching $\mathbf{v}$: these are the DEPENDENCY EQUATIONS.

Steps 4 and 5 work because row operations keep the relations among the columns (Theorem 4.7.6).

For the basis alone (Steps 1-3) any row echelon form will do; the dependency equations (Steps 4-5) are read off directly only from the REDUCED form.

EXAMPLE 10

Basis and dependency equations

$\mathbf{v}_1 = (1,-2,0,3)$, $\mathbf{v}_2 = (2,-5,-3,6)$, $\mathbf{v}_3 = (0,1,3,0)$, $\mathbf{v}_4 = (2,-1,4,-7)$, $\mathbf{v}_5 = (5,-8,1,2)$. Put them in the columns and row reduce:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

(a) The leading 1's are in columns 1, 2 and 4, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $\text{span}(S)$.

(b) In R: $\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{w}_5 = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_4$. The dependency equations are

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2, \qquad \mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$$

Check: $2(1,-2,0,3) - (2,-5,-3,6) = (0,1,3,0) = \mathbf{v}_3$.

Extending an independent set to a basis of $\mathbb{R}^n$

PROCEDURE To extend a linearly independent set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ to a basis of $\mathbb{R}^n$:

1) Form $A = [\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_k \;\; \mathbf{e}_1 \;\cdots\; \mathbf{e}_n\,]$.
2) Find its reduced row echelon form.
3) The columns of A in the positions of the leading 1's form a basis of $\mathbb{R}^n$ that contains S.

Example: extend $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathbf{v}_1 = (1,0,2,1)$, $\mathbf{v}_2 = (2,1,4,2)$, to a basis of $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Leading 1's in columns 1, 2, 3 and 5: the basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_3\}$.

The vectors of S come first and are independent, so they always get leading 1's; the $\mathbf{e}_j$ fill the gaps. This is Theorem 4.5.5(b) made concrete.

SECTION 4.8

Rank, Nullity, and the Fundamental Matrix Spaces

GOAL Measure the fundamental spaces. Their dimensions are the rank and the nullity, and one equation, $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$, ties them together. The rank also decides when $A\mathbf{x} = \mathbf{b}$ is consistent.

1. Row space and column space have the same dimension
2. Rank and nullity
3. The Dimension Theorem
4. Rank and consistency of $A\mathbf{x} = \mathbf{b}$
5. The four fundamental spaces
6. The Equivalence Theorem

Rank and nullity

In Examples 6 and 7 of Section 4.7, $\text{row}(A)$ and $\text{col}(A)$ both had 3 basis vectors. That is not an accident:

THEOREM 4.8.1 The row space and the column space of a matrix have the same dimension.

Proof

Let R be a row echelon form of A. By Theorems 4.7.4 and 4.7.6, $\dim\text{row}(A) = \dim\text{row}(R)$ and $\dim\text{col}(A) = \dim\text{col}(R)$. By Theorem 4.7.5 both equal the number of leading 1's of R. ■

DEFINITION 1 The common dimension of $\text{row}(A)$ and $\text{col}(A)$ is the RANK of A, written $\text{rank}(A)$. The dimension of $\text{null}(A)$ is the NULLITY of A, written $\text{nullity}(A)$.

So $\text{rank}(A)$ = the number of leading 1's in any row echelon form of A.

Example 2: the rows of an $m \times n$ matrix lie in $\mathbb{R}^n$ and its columns in $\mathbb{R}^m$, so $\text{rank}(A) \leq \min(m, n)$.

EXAMPLE 1

Rank and nullity of a 4×6 matrix

$$A=\begin{bmatrix}-1 & 2 & 0 & 4 & 5 & -3\\ 3 & -7 & 2 & 0 & 1 & 4\\ 2 & -5 & 2 & 4 & 6 & 1\\ 4 & -9 & 2 & -4 & -4 & 7\end{bmatrix} \xrightarrow{M_1^{(-1)},\, A_{1,2}^{(-3)},\, A_{1,3}^{(-2)},\, A_{1,4}^{(-4)}} \begin{bmatrix}1 & -2 & 0 & -4 & -5 & 3\\ 0 & -1 & 2 & 12 & 16 & -5\\ 0 & -1 & 2 & 12 & 16 & -5\\ 0 & -1 & 2 & 12 & 16 & -5\end{bmatrix}$$

$$\xrightarrow{M_2^{(-1)},\, A_{2,3}^{(1)},\, A_{2,4}^{(1)},\, A_{2,1}^{(2)}} \begin{bmatrix}1 & 0 & -4 & -28 & -37 & 13\\ 0 & 1 & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} = R$$

R has two leading 1's, so $\text{rank}(A)=2$. In $A\mathbf{x}=\mathbf{0}$ the leading variables are x_1, x_2, and $x_3=r$, $x_4=s$, $x_5=t$, $x_6=u$ are free:

$$\mathbf{x}=r(4,2,1,0,0,0)+s(28,12,0,1,0,0)+t(37,16,0,0,1,0)+u(-13,-5,0,0,0,1)$$

These four vectors form a basis for $\text{null}(A)$, so $\text{nullity}(A)=4$.

THEOREM 4.8.2

The Dimension Theorem for matrices

THEOREM 4.8.2 (Dimension Theorem for Matrices) If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Proof

The system $A\mathbf{x} = \mathbf{0}$ has n unknowns, and each unknown is either a leading variable or a free variable:

$$(\text{number of leading variables}) + (\text{number of free variables}) = n$$

There is one leading variable for each leading 1, so their number is $\text{rank}(A)$. The free variables are the parameters of the general solution, and the vectors that multiply them form a basis for $\text{null}(A)$ (Section 4.5), so their number is $\text{nullity}(A)$. ■

Example 3: in Example 1, $\text{rank}(A) + \text{nullity}(A) = 2 + 4 = 6$, the number of columns of A.

Count COLUMNS, not rows: n is the number of unknowns of $A\mathbf{x} = \mathbf{0}$.

Rank, nullity and the solutions of $A\mathbf{x} = \mathbf{0}$

THEOREM 4.8.3 If A is an $m \times n$ matrix, then

(a) $\text{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$;

(b) $\text{nullity}(A)$ = the number of parameters (free variables) in the general solution of $A\mathbf{x} = \mathbf{0}$.

Example 4

(a) A is 5×7 with rank 3. How many parameters does the general solution of $A\mathbf{x} = \mathbf{0}$ have?

$$\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4 \text{ parameters}$$

(b) A is 5×7, and the solution space of $A\mathbf{x} = \mathbf{0}$ has dimension 2. What is $\text{rank}(A)$?

$$\text{rank}(A) = n - \text{nullity}(A) = 7 - 2 = 5$$

EXAMPLE

Rank, nullity and a basis for the column space

For A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_5$: (a) find $\text{rank}(A)$ and $\text{nullity}(A)$; (b) find a basis for $\text{col}(A)$ made of columns of A; (c) write $\mathbf{a}_3$ as a combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{A_{1,3}^{(2)},\, M_2^{(-1)},\, A_{2,3}^{(1)},\, A_{2,4}^{(-3)}} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{I_{3,4},\, M_3^{(1/3)},\, A_{3,1}^{(-1)},\, A_{3,2}^{(1)}} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

(a) Three leading 1's: $\text{rank}(A) = 3$, and $\text{nullity}(A) = 5 - 3 = 2$ by the Dimension Theorem.

(b) Leading 1's in columns 1, 2, 4: $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $\text{col}(A)$.

(c) Column 3 of R is $(-2,3,0,0)$, so $\mathbf{a}_3 = -2\mathbf{a}_1 + 3\mathbf{a}_2$.

When is $A\mathbf{x} = \mathbf{b}$ consistent, and how many parameters?

CONSISTENCY AND RANK A system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\operatorname{rank}([\,A \mid \mathbf{b}\,]) = \operatorname{rank}(A)$.

Why

By Theorem 4.7.1, $A\mathbf{x} = \mathbf{b}$ is consistent exactly when $\mathbf{b} \in \operatorname{col}(A)$. The matrix $[\,A \mid \mathbf{b}\,]$ has the columns of A and one extra column $\mathbf{b}$:

- if $\mathbf{b} \in \operatorname{col}(A)$, the extra column adds nothing to the column space, so the two ranks are equal;
- if $\mathbf{b} \notin \operatorname{col}(A)$, it enlarges the column space by one dimension (Plus/Minus Theorem): $\operatorname{rank}([\,A \mid \mathbf{b}\,]) = \operatorname{rank}(A) + 1$.

THEOREM 4.8.4 If $A\mathbf{x} = \mathbf{b}$ is a consistent system of m equations in n unknowns and $\operatorname{rank}(A) = r$, then its general solution has $n - r$ parameters.

Why: the general solution is $\mathbf{x}_0 + \mathbf{x}_h$ (Theorem 4.7.2), and $\mathbf{x}_h$ has $\operatorname{nullity}(A) = n - r$ parameters.

EXAMPLE

Consistency from the ranks

Is this system consistent?

$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ x_1 + x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Plan: row reduce $[\,A \mid \mathbf{b}\,]$ and compare $\operatorname{rank}(A)$ with $\operatorname{rank}([\,A \mid \mathbf{b}\,])$.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-1)},\, A_{1,3}^{(-3)}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 4 \\ 0 & -1 & 2 & 4 \end{array}\right] \xrightarrow{M_2^{(-1)},\, A_{2,3}^{(1)},\, A_{2,1}^{(-1)}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $\operatorname{rank}(A) = 2$ and $\operatorname{rank}([\,A \mid \mathbf{b}\,]) = 2$: two nonzero rows either way. The system is CONSISTENT, with $n - r = 3 - 2 = 1$ parameter: $x_3 = t$, $x_1 = 3 - t$, $x_2 = -4 + 2t$.
- Column view: in the reduced matrix the last column is $3\,(\text{column } 1) - 4\,(\text{column } 2)$, so $\mathbf{b} = 3\mathbf{c}_1 - 4\mathbf{c}_2 + 0\,\mathbf{c}_3 \in \operatorname{col}(A)$ (Theorem 4.7.6).

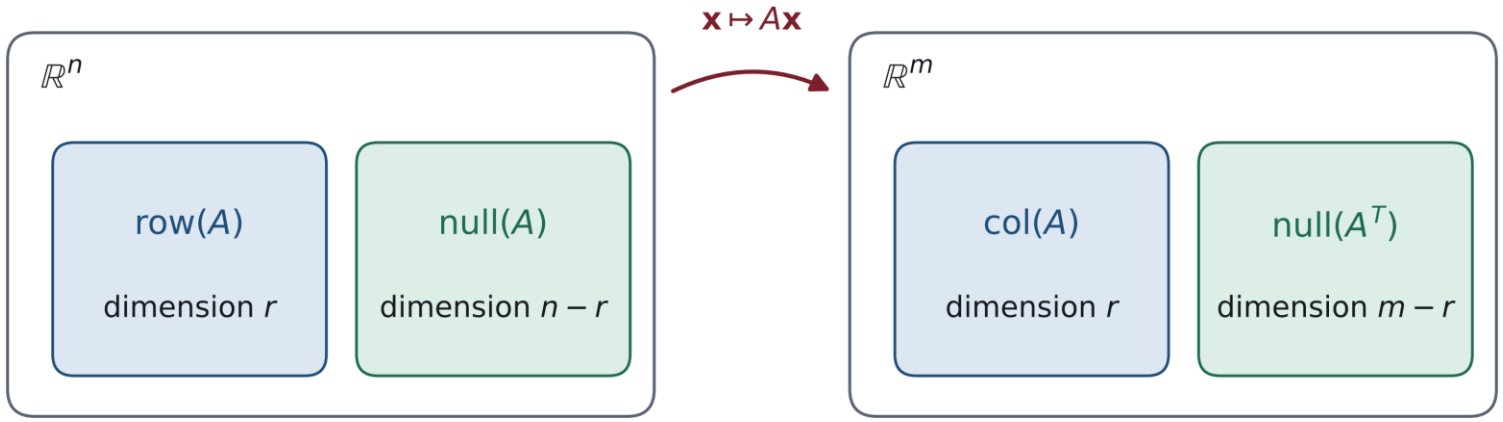
> An inconsistent system shows up as a row $[\,0 \;\; 0 \;\; \cdots \;\; 0 \mid c\,]$ with $c \neq 0$: that row raises the rank of $[\,A \mid \mathbf{b}\,]$ by one.

The four fundamental spaces and their dimensions

A and A^T give six spaces, but $\text{row}(A^T) = \text{col}(A)$ and $\text{col}(A^T) = \text{row}(A)$. The four FUNDAMENTAL SPACES of A are $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$ and $\text{null}(A^T)$.

THEOREM 4.8.5 $\text{rank}(A^T) = \text{rank}(A)$. Proof: $\text{rank}(A) = \text{dimrow}(A) = \text{dimcol}(A^T) = \text{rank}(A^T)$. ■

A^T has m columns, so the Dimension Theorem for A^T gives $\text{rank}(A) + \text{nullity}(A^T) = m$. For an $m \times n$ matrix of rank r:



EXAMPLE

What the ranks tell you

From the size of A, $\operatorname{rank}(A)$ and $\operatorname{rank}([\,A \mid \mathbf{b}\,])$ alone:

	(a)	(b)	(c)	(d)
Size of A	3×3	3×3	5×9	4×4
$\operatorname{rank}(A) = r$	3	2	2	0
$\operatorname{rank}([\,A \mid \mathbf{b}\,])$	3	3	2	0
$\dim\operatorname{row}(A) = \dim\operatorname{col}(A) = r$	3	2	2	0
$\dim\operatorname{null}(A) = n - r$	0	1	7	4
$\dim\operatorname{null}(A^T) = m - r$	0	1	3	4
Consistent?	yes	no	yes	yes
Parameters $n - r$	0	no solution	7	4

In (b), $\mathbf{b} \notin \operatorname{col}(A)$; in (d), A is the zero matrix, so $\mathbf{b} = \mathbf{0}$ and every $\mathbf{x} \in \mathbb{R}^4$ is a solution.

THEOREM 4.8.8

The Equivalence Theorem, so far

THEOREM 4.8.8 For an $n \times n$ matrix A, the following statements are equivalent:

From Chapters 1 and 2

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$.

(g) $\det(A) \neq 0$.

New in Chapter 4

(h) The columns of A are linearly independent.

(i) The rows of A are linearly independent.

(j) The columns of A span $\mathbb{R}^n$.

(k) The rows of A span $\mathbb{R}^n$.

(l) The columns of A form a basis for $\mathbb{R}^n$.

(m) The rows of A form a basis for $\mathbb{R}^n$.

(n) $\text{rank}(A) = n$.

(o) $\text{nullity}(A) = 0$.

New links: (b) ⇒ (o): no parameters; (o) ⇒ (n): Dimension Theorem; (n) ⇒ (b): n leading variables, so no free ones.

Anton also lists (p) and (q), about orthogonal complements; this course does not cover them.

SUMMARY

Unit 3C in one slide

4.7 Row space, column space, null space

- $A\mathbf{x} = \mathbf{b}$ is consistent exactly when $\mathbf{b} \in \text{col}(A)$; then every solution is $\mathbf{x}_0 + \mathbf{x}_h$ with $\mathbf{x}_h \in \text{null}(A)$.
- Row operations keep $\text{row}(A)$, $\text{null}(A)$ and the relations among the columns; they can change $\text{col}(A)$.
- With R the reduced row echelon form of A: the nonzero rows of R give a basis for $\text{row}(A)$; the columns of A at the leading 1's give a basis for $\text{col}(A)$, and the other columns of R give the dependency equations; the vectors that multiply the free parameters give a basis for $\text{null}(A)$.

4.8 Rank and nullity

- $\text{rank}(A) = \text{dimrow}(A) = \text{dimcol}(A) = \text{rank}(A^T)$, and $\text{rank}(A) + \text{nullity}(A) = n$, the number of columns.
- For an $m \times n$ matrix of rank r: $\text{dimnull}(A) = n - r$ and $\text{dimnull}(A^T) = m - r$.
- $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\text{rank}([\,A \mid \mathbf{b}\,]) = \text{rank}(A)$, and then its general solution has $n - r$ parameters.
- For an $n \times n$ matrix: A is invertible $\Leftrightarrow \text{rank}(A) = n \Leftrightarrow \text{nullity}(A) = 0$.