

# 7 Cosets Tutorial

**Exercise 1:** Let  $H = \{0, \pm 3, \pm 6, \pm 9, \dots\} = 3\mathbb{Z}$ . Find all the left cosets of  $H$  in  $\mathbb{Z}$ . Let  $n$  be a positive integer. Let  $H = \{0, \pm n, \pm 2n, \pm 3n, \dots\} = n\mathbb{Z}$ . Find all left cosets of  $H$  in  $\mathbb{Z}$ . How many are there?

**Solution:** For  $3\mathbb{Z}$ :  $0 + 3\mathbb{Z}$ ,  $1 + 3\mathbb{Z}$ ,  $2 + 3\mathbb{Z}$ . In general,  $n\mathbb{Z}$  has exactly  $n$  cosets:  $k + n\mathbb{Z}$  for  $k = 0, 1, \dots, n - 1$ .

**Key idea:** Cosets of a subgroup of  $\mathbb{Z}$  are residue classes.

---

**Exercise 2:** Rewrite the condition  $a^{-1}b \in H$  (from property 6 of the lemma in this chapter) in additive notation. Assume the group is abelian.

**Solution:**  $a^{-1}b \in H \iff (-a) + b \in H \iff b - a \in H$ .

**Key idea:** Inverse  $\mapsto$  negative; product  $\mapsto$  sum.

**Exercise 3:** Let  $H$  be as in Exercise 1. Use Exercise 2 to decide whether the following cosets of  $H$  are the same: (a)  $11 + H$  and  $17 + H$ , (b)  $-1 + H$  and  $5 + H$ , (c)  $7 + H$  and  $23 + H$ .

**Solution:**  $11 - 17 = -6 \in 3\mathbb{Z} \Rightarrow 11 + H = 17 + H$ .  $-1 - 5 = -6 \in 3\mathbb{Z} \Rightarrow -1 + H = 5 + H$ .  $7 - 23 = -16 \notin 3\mathbb{Z} \Rightarrow 7 + H \neq 23 + H$ .

**Key idea:** Cosets are equal  $\Leftrightarrow$  the difference of representatives lies in  $H$ .

---

**Exercise 4:** Find all of the left cosets of  $\{1, 11\}$  in  $U(30)$ .

**Solution:**  $U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}$ ,  $H = \{1, 11\}$ . Cosets:  $1H = \{1, 11\}$ ,  $7H = \{7, 17\}$ ,  $13H = \{13, 23\}$ ,  $19H = \{19, 29\}$ .

**Key idea:** Multiply every element of the subgroup by a representative.

**Exercise 5:** Let  $a$  belong to a group and  $|a| = 30$ . How many left cosets of  $\langle a^5 \rangle$  in  $\langle a \rangle$  are there? List them. Do the same for  $\langle a^4 \rangle$  in  $\langle a \rangle$ .

**Solution:**  $|\langle a^5 \rangle| = 30 / \gcd(5, 30) = 6 \Rightarrow 5$  cosets:  $a^i \langle a^5 \rangle$  for  $i = 0, 1, 2, 3, 4$ .

$|\langle a^4 \rangle| = 30 / \gcd(4, 30) = 15 \Rightarrow 2$  cosets:  $\langle a^4 \rangle$  and  $a \langle a^4 \rangle$ .

**Key idea:** Index equals order quotient.

---

**Exercise 6:** Give a group  $G$  and subgroups  $H, K \leq G$  such that  $HK = \{hk : h \in H, k \in K\}$  is not a subgroup of  $G$ .

**Solution:**  $G = S_3, H = \langle (1\ 2) \rangle, K = \langle (1\ 3) \rangle$ . Then  $HK = \{e, (1\ 2), (1\ 3), (1\ 3\ 2)\}$  has 4 elements, and  $4 \nmid 6$ , so  $HK$  is not a subgroup.

**Key idea:**  $HK$  need not equal  $KH$ ; closure can fail.

**Exercise 8:** Let  $a, b \in G$  and  $H, K \leq G$ . If  $aH = bK$ , prove  $H = K$ .

**Solution:**  $aH = bK \Rightarrow H = a^{-1}bK$ . A single left translate of  $K$  equals  $H$ , hence  $H = K$ .

**Key idea:** One equal coset forces the subgroups to coincide.

---

**Exercise 9:** If  $H, K \leq G$  and  $g \in G$ , show that  $g(H \cap K) = gH \cap gK$ .

**Solution:**  $x \in g(H \cap K) \iff x = gy$  with  $y \in H \cap K \iff x \in gH$  and  $x \in gK \iff x \in gH \cap gK$ .

**Key idea:** Element chase; left multiplication respects intersections.

**Exercise 13:** Let  $G$  be a group of order 60. What are the possible orders for subgroups of  $G$ ?

**Solution:** The positive divisors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.

**Key idea:** Lagrange's theorem: subgroup order divides group order.

---

**Exercise 14:** Suppose  $K$  is a proper subgroup of  $H$  and  $H$  a proper subgroup of  $G$ . If  $|K| = 42$  and  $|G| = 420$ , what are the possible orders of  $H$ ?

**Solution:**  $|G : K| = 10$ . Intermediate indices divide 10, so  $|G : H| \in \{1, 2, 5, 10\}$  and  $|H| \in \{420, 210, 84, 42\}$ . Exclude 42 and 420 (proper containments)  $\Rightarrow |H| = 210$  or 84.

**Key idea:** Tower law  $|G : K| = |G : H| \cdot |H : K|$ .

**Exercise 15:** Let  $|G| = pq$  where  $p, q$  are prime. Prove that every proper subgroup of  $G$  is cyclic.

**Solution:** Proper subgroup orders are  $1, p, q$ , all prime  $\Rightarrow$  each subgroup is cyclic.

**Key idea:** Groups of prime order are cyclic.

---

**Exercise 16:** For  $n > 1$ ,  $\varphi(n)$  counts integers  $< n$  and coprime to  $n$ . Prove: if  $\gcd(a, n) = 1$ , then  $a^{\varphi(n)} \equiv 1 \pmod{n}$ .

**Solution:**  $a \in U(n) \Rightarrow |a| \mid \varphi(n) \Rightarrow a^{\varphi(n)} = 1$  in  $U(n)$ .

**Key idea:** Euler's theorem: the order of  $a$  divides  $|U(n)| = \varphi(n)$ .

**Exercise 17:** Compute  $5^{15} \pmod{7}$  and  $7^{13} \pmod{11}$ .

**Solution:**  $5^6 \equiv 1 \pmod{7} \Rightarrow 5^{15} = 5^{6 \cdot 2 + 3} \equiv 5^3 = 125 \equiv 6 \pmod{7}$ .  $7^{10} \equiv 1 \pmod{11} \Rightarrow 7^{13} \equiv 7^3 = 343 \equiv 1 \pmod{11}$ .

**Key idea:** Fermat's little theorem: reduce exponents modulo  $p - 1$ .

---

**Exercise 20:** Suppose  $H, K \leq G$ . If  $|H| = 12$  and  $|K| = 35$ , find  $|H \cap K|$ . Generalize.

**Solution:**  $|H \cap K| \mid \gcd(12, 35) = 1 \Rightarrow |H \cap K| = 1$ . In general,  $|H \cap K| \mid \gcd(|H|, |K|)$ .

**Key idea:** Intersection size divides the gcd of the orders.

**Exercise 21:** For any integer  $n \geq 3$ , prove that  $D_n$  has a subgroup of order 4 iff  $n$  is even.

**Solution:** If  $n$  is even,  $\langle r^{n/2}, s \rangle \cong V_4$ . If  $n$  is odd, there is no element of order 4 and hence no  $V_4$ .

**Key idea:** Existence of a half-turn (order 2 rotation) plus a reflection requires even  $n$ .

---

**Exercise 23:** Suppose  $G$  is abelian with an odd number of elements. Show that the product of all elements of  $G$  is the identity.

**Solution:** Pair each  $g$  with  $g^{-1}$ ; in odd order, the only self-inverse is  $e$ . Thus  $\prod_{g \in G} g = e$ .

**Key idea:** Pairing with inverses cancels all non-identity elements.

**Exercise 24:** Prove that every group of order 4 is abelian.

**Solution:** If some element has order 4, the group is cyclic  $\Rightarrow$  abelian. Otherwise all non-identity elements have order 2  $\Rightarrow V_4$ , abelian.

**Key idea:** The only possibilities are  $C_4$  or  $V_4$ .

---

**Exercise 27:** Let  $|G| = 33$ . What are the possible element orders? Show  $G$  has an element of order 3.

**Solution:** Divisors of 33: 1, 3, 11, 33. By Cauchy, there are elements of orders 3 and 11.

**Key idea:** Cauchy gives an element for each prime dividing  $|G|$ .

**Exercise 28:** Suppose  $G$  is non-abelian with  $|G| = 8$ . Prove that  $G$  has an element of order 4.

**Solution:** If all non-identity elements had order 2, then  $G$  would be abelian—a contradiction. Hence some element has order 4.

**Key idea:** Excluding order 4 forces commutativity.

---

**Exercise 29:** Prove that any group of order 55 must have exactly one subgroup of order 5 or exactly 11 subgroups of order 5.

**Solution:** By Sylow,  $n_5 \equiv 1 \pmod{5}$  and  $n_5 \mid 11$ , so  $n_5 \in \{1, 11\}$ .

**Key idea:** Sylow congruence and divisibility conditions pin down the count.

**Exercise 35:** Let  $H \subseteq K \subseteq G$  be subgroups of a finite group  $G$ . Prove that  $|G : H| = |G : K| \cdot |K : H|$ .

**Solution:**  $|G : H| = \frac{|G|}{|H|} = \frac{|G|}{|K|} \cdot \frac{|K|}{|H|} = |G : K| \cdot |K : H|$ .

**Key idea:** Indices multiply like fractions.

---

**Exercise 38:** Let  $G$  be a group with  $|G| = 21$ . If  $g \in G$  and  $g^{14} = e$ , what are the possibilities for  $|g|$ ?

**Solution:**  $|g| \mid \gcd(21, 14) = 7 \Rightarrow |g| \in \{1, 7\}$ .

**Key idea:** The order divides any exponent yielding the identity.

**Exercise 50:** Prove that an abelian group of order 20 has an element of order 5. Does your proof generalize to show every abelian group of order  $4p^n$  (with  $p$  prime) has a subgroup of order  $p$ ?

**Solution:** Since  $5 \mid 20$ , Cauchy gives an element of order 5. For  $|G| = 4p^n$  with  $p$  odd prime, Cauchy again gives an element (hence cyclic subgroup) of order  $p$ .

**Key idea:** Apply Cauchy's theorem.

---

**Exercise 60:** Prove that  $A_5$  has a subgroup of order 12.

**Solution:** The stabilizer of a point in  $A_5$  is isomorphic to  $A_4$ , which has order 12.

**Key idea:** Use point stabilizers in the natural action.

**Exercise 61:** Prove that  $A_5$  has no subgroup of order 30.

**Solution:** A subgroup of index 2 would be normal, but  $A_5$  is simple. Contradiction.

**Key idea:** Simplicity of  $A_5$  rules out index-2 subgroups.

---

**Exercise 69:** Let  $G =$

$\{(1), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}$ .

(a) Find the stabilizer of 1 and the orbit of 1. (b) Find the stabilizer of 3 and the orbit of 3.

(c) Find the stabilizer of 5 and the orbit of 5.

**Solution:**  $\text{Stab}_G(1) = \{(1), (56)(13)\}$ ,  $\text{Orb}_G(1) = \{1, 3\}$ .  $\text{Stab}_G(3) = \{(1), (56)(13)\}$ ,  $\text{Orb}_G(3) = \{3, 1\}$ .  $\text{Stab}_G(5)$  has order 4;  $\text{Orb}_G(5) = \{5, 6\}$ .

**Key idea:** Cycle notation reveals fixed points and swaps.

**Exercise 71:** Let  $G = \text{GL}(2, \mathbb{R})$  and  $H = \text{SL}(2, \mathbb{R})$ . Let  $A \in G$  with  $\det A = 2$ . Prove that  $AH$  is the set of all  $2 \times 2$  matrices in  $G$  having determinant 2.

**Solution:** For  $X \in AH$ ,  $\det X = \det A \cdot \det h = 2$ . Conversely, if  $\det M = 2$ , then  $M = A(A^{-1}M)$  with  $A^{-1}M \in H$ . Hence  $AH = \{M \in \text{GL}(2, \mathbb{R}) : \det M = 2\}$ .

**Key idea:** Determinant multiplies; coset equals the determinant-2 level set.