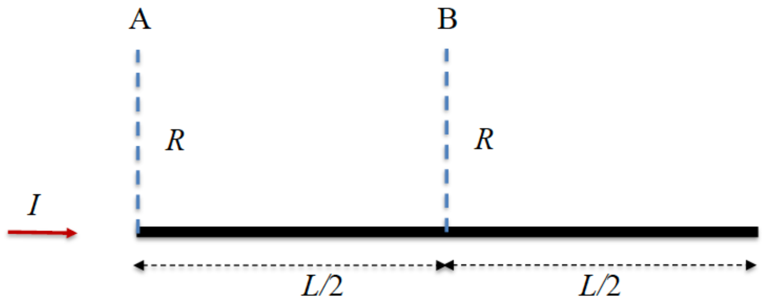
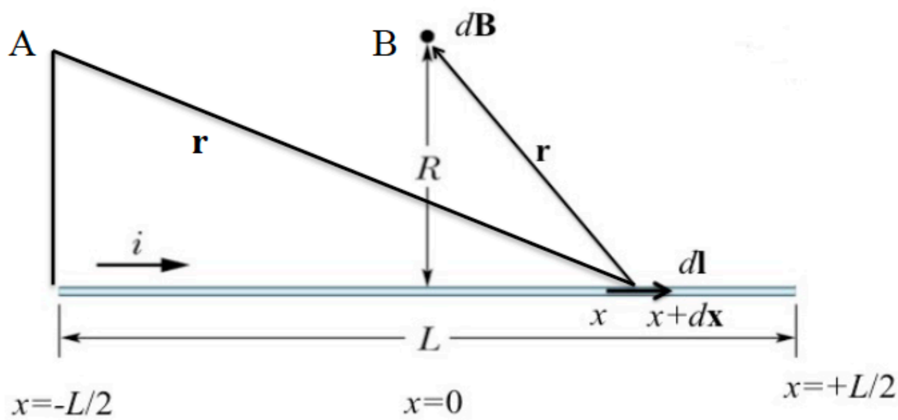
PHYSICS 507
6th HOMEWORK
Prof. V. Lempesis
Hand in: Tuesday 30th of March 2021, time: 23:59

1. Find the magnetic field created by the following finite length current-carrying wire at points A and B.



Solution:

Let us first calculate the magnetic field at point B. If we consider an element of the wire



To solve this problem we need to apply Biot-Savart Law. We consider the elementary part $d\mathbf{l}$ of the wire at a position x having length dx. Thus $d\mathbf{l} = dx\hat{\mathbf{x}}$. This part is flown by a current I so at the point B it creates a magnetic field $d\mathbf{B}$ given by:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$

where $\mathbf{r}$ is a vector having its tail (beginning) at the tail of $d\mathbf{l}$ and its tip (end) at the point B. Thus $\mathbf{r} = (0, h, 0) - (x, 0, 0)$ or $\mathbf{r} = (-x, h, 0)$. Then

$$d\mathbf{l}\times\mathbf{r}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ dx & 0 & 0\\ -x & R & \end{vmatrix}=Rdx\hat{\mathbf{z}}.$$

The magnitude of r is given by $r=\left(x^2+R^2\right)^{1/2}$. Thus for the elementary magnetic field we have:

$$d\mathbf{B}=\frac{\mu_0 I}{4\pi}\cdot\frac{Rdx}{\left(x^2+R^2\right)^{3/2}}\hat{\mathbf{z}}.$$

The total magnetic field is taken by an integration we get

$$\mathbf{B}=\int_{-L/2}^{+L/2}d\mathbf{B}=\hat{\mathbf{z}}\frac{\mu_0 IR}{4\pi}\cdot\int_{-L/2}^{L/2}\frac{dx}{\left(x^2+R^2\right)^{3/2}}=\hat{\mathbf{z}}\frac{\mu_0 IR}{4\pi}\cdot\left[\frac{x}{R^2\left(x^2+R^2\right)^{1/2}}\right]_{-L/2}^{+L/2}$$

$$\mathbf{B}=\hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\cdot\frac{L}{R\left(\left(L/2\right)^2+R^2\right)^{1/2}}$$

$$\mathbf{B}=\hat{\mathbf{z}}\frac{\mu_0 I}{2\pi R}\left\{\frac{L}{\sqrt{L^2+4R^2}}\right\}$$

Similarly for point A we have that $\mathbf{r}=\left(-\frac{L}{2}-x,\ R,0\right)$. Thus,

$$d\mathbf{l}\times\mathbf{r}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ dx & 0 & 0\\ -\frac{L}{2}-x & R & 0\end{vmatrix}=Rdx\hat{\mathbf{z}}$$

So

$$d\mathbf{B}=\frac{\mu_0 I}{4\pi}\frac{Rdx}{\left[\left(\frac{L}{2}+x\right)^2+R^2\right]^{3/2}}\hat{\mathbf{z}}$$

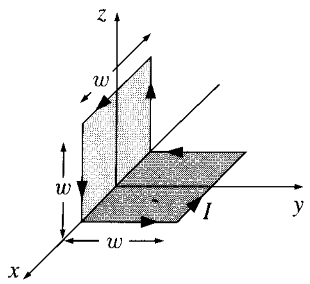
$$\mathbf{B}=\hat{\mathbf{z}}\frac{\mu_0 IR}{4\pi}\int_{-L/2}^{+L/2}\frac{1}{\left[\left(\frac{L}{2}+x\right)^2+R^2\right]^{3/2}}dx$$

$$= \hat{\mathbf{z}}\frac{\mu_0 IR}{4\pi R^2}\left\{\frac{2x+L}{\sqrt{L^2+4xL+4R^2+4x^2}}\Big|_{-L/2}^{L/2}\right\}$$

$$\hat{\mathbf{z}}\frac{\mu_0 I}{4\pi R}\left\{\frac{2L/2+L}{\sqrt{L^2+4LL/2+4R^2+4L^2/4}}-\frac{-2L/2+L}{\sqrt{L^2-4LL/2+4R^2+4L^2/4}}\right\}$$

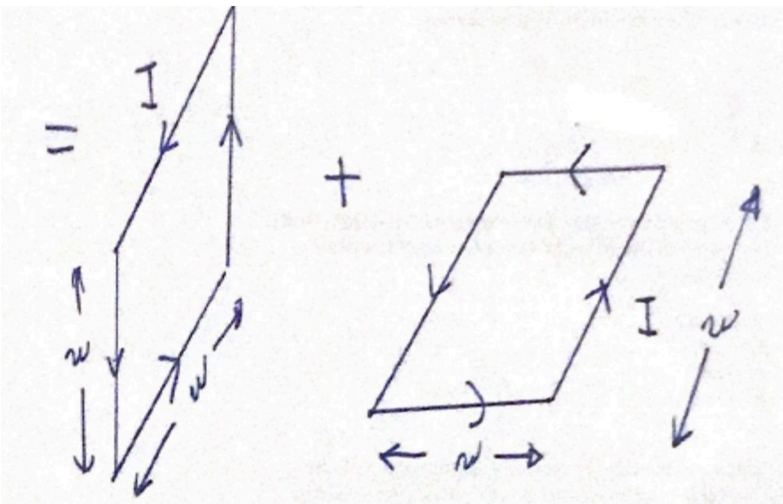
$$= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi R}\left\{\frac{L}{\sqrt{L^2+R^2}}\right\}$$

2. Find the magnetic dipole moment of the loop shown in figure below. All sides have length w, and it carries a current I.



Solution:

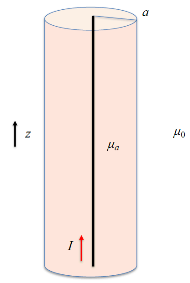
The superposition principle implies that the magnetic dipole moment will be the resultant of the dipole moments of the two frames. Thus:



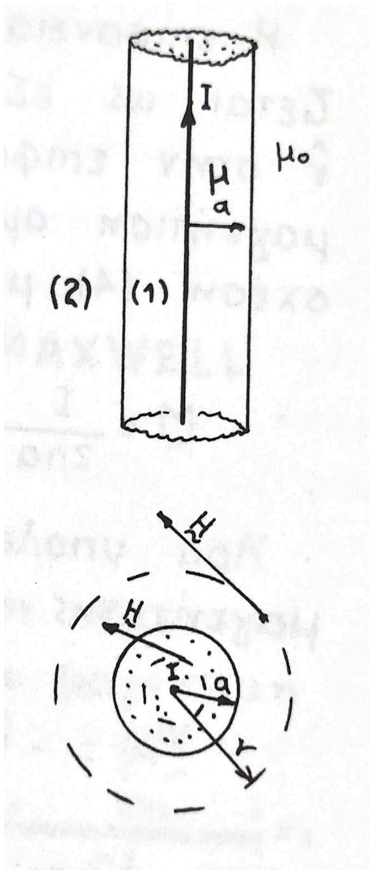
$$\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 = Iw^2\hat{\mathbf{y}} + Iw^2\hat{\mathbf{z}}$$

Thus $m = Iw^2\sqrt{2}$ and at a direction on the y-z plane at 45^0 with respect to the positive part of the y-axis.

3. A long and thin wire is flown by a current I along the z-direction. The wire is enclosed by a cylindrical magnetic material of radius a and magnetic permeability μ_a. Find at all space points the vectors **H**, **B**, **M** and the volume and surface current densities associated with **M**.



Solution:



The presence of the magnetic material forces us to work with the auxiliary field **H**. The problem has an obvious cylindrical symmetry. Assume a loop of radius r centered at the wire, then the field **H** is tangent to the loop and we have

$$H2\pi r = I \Rightarrow \mathbf{H} = \frac{I}{2\pi r}\hat{\boldsymbol{\phi}}$$

Since I is the only current of free charges. For this, the relation above holds in all space both for $r > a$ and for $r < a$.

The magnetic field **B** is given by:

$$\mathbf{B}_1 = \frac{\mu_a I}{2\pi r}\widehat{\boldsymbol{\phi}}, \quad r < a$$

$$\mathbf{B}_2 = \frac{\mu_0 I}{2\pi r}\widehat{\boldsymbol{\phi}}, \quad r > a$$

The magnetization inside the material is:

$$\mathbf{M}_1 = \frac{\mathbf{B}_1}{\mu_0} - \mathbf{H}_1 = \frac{\mu_a I}{\mu_0 2\pi r}\widehat{\boldsymbol{\phi}} - \frac{I}{2\pi r}\widehat{\boldsymbol{\phi}} = \frac{I}{2\pi r}\left(\frac{\mu_a}{\mu_0} - 1\right)\widehat{\boldsymbol{\phi}}, \quad r < a$$

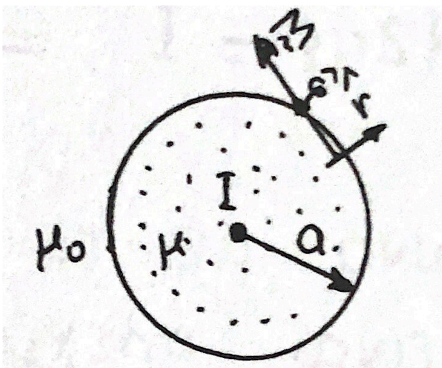
$$\mathbf{M}_2 = \frac{\mathbf{B}_2}{\mu_0} - \mathbf{H}_1 = \frac{\mu_0 I}{\mu_0 2\pi r}\widehat{\boldsymbol{\phi}} - \frac{I}{2\pi r}\widehat{\boldsymbol{\phi}} = 0, \quad r > a$$

The volume current density associated with **M** is given by:

$$\mathbf{J_M} = \vec{\nabla} \times \mathbf{M}_1$$

Using the expression for curl in cylindrical coordinates we can show that $\mathbf{J_M} = 0$.

The surface current density $\mathbf{K_M}$ associated with **M** is found as follows: we consider the unitary vector $\hat{\mathbf{r}}$ perpendicular to the surface of the material and with a direction outwards.



The magnetization right after inside the material surface is given by:

$$\mathbf{M} = \frac{I}{2\pi r}\left(\frac{\mu_a}{\mu_0} - 1\right)\Bigg|_{r=a}\widehat{\boldsymbol{\phi}} = \frac{I}{2\pi a}\left(\frac{\mu_a}{\mu_0} - 1\right)\widehat{\boldsymbol{\phi}}$$

So for $\mathbf{K_M}$ we have:

$$\mathbf{K_M} = \mathbf{M} \times \hat{\mathbf{r}} = -\frac{I}{2\pi a}\left(\frac{\mu_a}{\mu_0} - 1\right)\hat{\mathbf{z}}$$