4 Vector Spaces continued

4.3 Linear Independence

4.4 Coordinates and Basis

4.5 Dimension

4.6 Change of Basis

4.3 Linear Independence

Definition Linear Independence (L.I.) and Linear Dependence (L.D.)

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$: a set of vectors in a vector space V

For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$

(1) If the equation has only the trivial solution $(c_1 = c_2 = \cdots = c_k = 0)$
then S (or $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$) is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all c_i's are zero),
then S (or $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$) is called linearly dependent.

Example Testing For Linear Independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\{(1,\ 2,\ 3),(0,\ 1,\ 2),(-2,\ 0,\ 1)\}$$

Solution

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}\ \Rightarrow\ \begin{aligned} c_1 \qquad\quad -2c_3 &= 0\\ 2c_1+\ c_2+\qquad\ &= 0\\ 3c_1+2c_2+\ c_3 &= 0\end{aligned}$$

$$\Rightarrow\left[\begin{array}{ccc|c}1&0&-2&0\\2&1&0&0\\3&2&1&0\end{array}\right]\xrightarrow{\text{G.-J. E.}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

$$\Rightarrow c_1=c_2=c_3=0\quad(\text{only the trivial solution})$$

$$(\text{or }\det(A)=-1\neq 0,\ \text{so there is only the trivial solution})$$

Example Testing For Linear Independence

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$$

Solution

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

i.e., $c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1 + 2c_2 \quad\; &= 0 \\ c_1 + 5c_2 + c_3 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G. E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2=-1$, $c_3=3$)

$\Rightarrow$ S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

Example Testing For Linear Independence

Determine whether the following set of vectors in $M_{2\times2}$ is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}2&1\\0&1\end{bmatrix},\begin{bmatrix}3&0\\2&1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix}\right\}$$

Solution

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0} \quad\Rightarrow\quad c_1\begin{bmatrix}2&1\\0&1\end{bmatrix}+c_2\begin{bmatrix}3&0\\2&1\end{bmatrix}+c_3\begin{bmatrix}1&0\\2&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow\quad \begin{aligned}2c_1+3c_2+c_3&=0\\ c_1\qquad\qquad&=0\\ 2c_2+2c_3&=0\\ c_1+c_2\qquad&=0\end{aligned} \quad\Rightarrow\quad \left[\begin{array}{ccc|c}2&3&1&0\\1&0&0&0\\0&2&2&0\\1&1&0&0\end{array}\right]\xrightarrow{\text{G.-J. E.}}\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$\Rightarrow\ c_1=c_2=c_3=0$ (This system has only the trivial solution) $\Rightarrow$ S is linearly independent

Theorem A Defining Property of L.D.

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, for $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ in S can be written as a linear combination of the other vectors in S.

Proof. $(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_i\mathbf{v}_i + \ldots + c_k\mathbf{v}_k = \mathbf{0}$

$\because S$ is linearly dependent (there exist nontrivial solutions) $\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \left(-\frac{c_1}{c_i}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{i-1}}{c_i}\right)\mathbf{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right)\mathbf{v}_{i+1} + \cdots + \left(-\frac{c_k}{c_i}\right)\mathbf{v}_k$$

$(\Leftarrow)$ Let $\mathbf{v}_i = d_1\mathbf{v}_1 + \ldots + d_{i\text{-}1}\mathbf{v}_{i\text{-}1} + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k$

$$\Rightarrow d_1\mathbf{v}_1 + \ldots + d_{i\text{-}1}\mathbf{v}_{i\text{-}1} + (-1)\mathbf{v}_i + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k = \mathbf{0}$$

$\Rightarrow c_1 = d_1\ ,\ c_2 = d_2\ , \ldots,\ c_i = -1\ , \ldots,\ c_k = d_k$ (there exits at least this nontrivial solution)

$\Rightarrow S$ is linearly dependent

Theorem Easily Checked Sets

(a) A finite set that contains 0 is linearly dependent.
(b) A set with exactly one vector is linearly independent if and only if that vector is not 0.
(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Proof.

(a) $S = \{0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is L.D. since $1 \cdot 0 + 0v_1 + 0v_2 + \cdots + 0v_k = 0$.

(b) Follows from the fact that $cv = 0 \Leftrightarrow c = 0 \; or \; v = 0$.

(c) Follows from the previous theorem and is left as an exercise.

Theorem Checking L.D. From Number of Vectors

Let $S = \{v_1, v_2, \cdots, v_r\} \subseteq R^n$. If $r > n$, then S is L.D..

Proof The vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_r v_r = 0$ gives a homogeneous linear System of n equations in r variables. Since $r > n$, it must have a nontrivial solution. So, the set S must be L.D..

Exercise:

True or False?

A set containing a single vector is linearly independent.

A) True.

B) False.

Multiple Choice

Exercise:

Which of the following sets is linearly independent?

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -12 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Multiple Choice

Exercise:

True or False?

The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

A) True.

B) False.

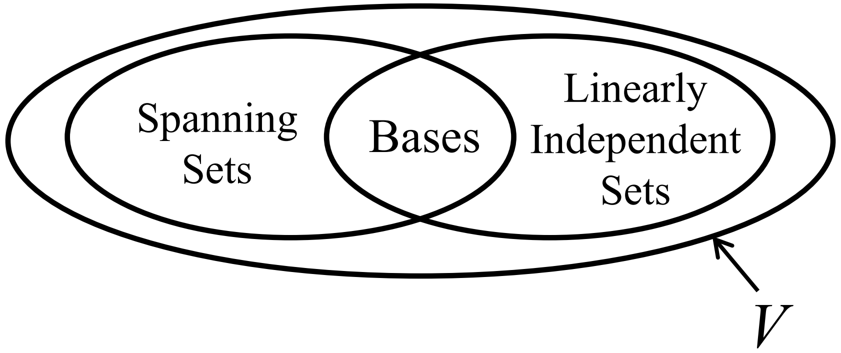
Multiple Choice

4.4 Coordinates and Basis
4.5 Dimension
4.6 Change of Basis

Definition Basis

$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$: a set of vectors in a vector space V is called a basis if

(1) S is linearly independent, and
(2) S spans V, i.e., $span(S) = V$.



Note: A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S.

Example A (Standard) Basis For R^3

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for R^3:

(1) S is L.I. for if $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$, then $c_1 = c_2 = c_3 = 0$.

(2) span$(S) = V$, since $(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$.

Example A (Standard) Basis For R^n

$$S = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}, \text{where } \boldsymbol{e}_1 = (1,0,\dots,0), \boldsymbol{e}_2 = (0,1,\dots,0), \dots, \boldsymbol{e}_n = (0,0,\dots,1)$$

Example A (Standard) Basis For $M_{m\times n}$

$$S=\{ E_{ij} \mid 1\le i\le m,\ 1\le j\le n \}, \text{ and in } E_{ij} \begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$$

E.g. For $M_{2\times 2}$ we have:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Example A (Standard) Basis For P_n

$S = \{1, x, x^2, \ldots, x^n\}$ E.g. for $P_3(x)$: $S = \{1, x, x^2, x^3\}$

Example A Nonstandard Basis For R^2

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,1),\ (1,-1)\}$ is a basis for R^2

Solution

(1) For any $\mathbf{u} = (u_1, u_2) \in R^2$, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{u} \Rightarrow \begin{cases} c_1 + c_2 = u_1 \\ c_1 - c_2 = u_2 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, the system has a unique solution for each **u**. Thus S spans R^2

(2) For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$

Because the coefficient matrix of this system has a **nonzero determinant**, we know that the system has only the trivial solution. Thus S is linearly independent

From (1) and (2) we conclude that S is a (nonstandard) basis for R^2

Theorem Uniqueness of Basis Representation of Vectors

Let $S = \{v_1, v_2, \cdots, v_n\}$ be a basis for a vector space V, then every vector v in V can be written in one and only one way as a linear combination of vectors in S.

Proof

$$\because S \text{ is a basis} \Rightarrow \begin{cases} (1)\ \text{span}(S) = V \\ (2)\ S \text{ is linearly independent} \end{cases}$$

$$\because \text{span}(S) = V \quad \text{Let} \quad \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n$$

$$\Rightarrow \mathbf{v} + (-1)\mathbf{v} = \mathbf{0} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \ldots + (c_n - b_n)\mathbf{v}_n$$

$$\because S \text{ is linearly independent} \Rightarrow \text{with only the trivial solution}$$

$$\Rightarrow \text{coefficients for } \mathbf{v}_i \text{ are all zero}$$

$$\Rightarrow c_1 = b_1, c_2 = b_2, \ldots, c_n = b_n \text{ (i.e., unique basis representation)}$$

Theorem Sets larger than a basis must be dependent

Let $S = \{v_1, v_2, \cdots, v_n\}$ be a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent. (In other words, every linearly independent set contains at most n vectors)

Proof

$$\text{Let } S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \ , m > n$$

$$\because \text{span}(S) = V$$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Consider $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

(if k_i's are not all zero, S_1 is linearly dependent)

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ $(d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m)$

$\because S$ is L.I. $\Rightarrow d_i=0 \quad \forall i$ i.e.,

$$\begin{aligned} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m &= 0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m &= 0 \end{aligned}$$

$\because$ This is a homogeneous system with more variables ($k_1, k_2, \ldots, k_m$), than equations (n equations), then it must have infinitely many solutions

$\Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial (nonzero) solution $\Rightarrow S_1$ is L.D.

Note: This theorem tells us that if V has a bas with n vectors then every independent set must have at most n vectors.

Theorem Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Proof Using the note in the previous slide:

$$\left.\begin{array}{l} S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \\ S' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \end{array}\right\} \text{are two bases for } V$$

$$\left.\begin{array}{l} \left.\begin{array}{c} S \text{ is a basis } V \\ S' \text{ is a set of L.I. vectors} \end{array}\right\} \Rightarrow m \le n \\ \left.\begin{array}{c} S' \text{ is a basis } V \\ S \text{ is a set of L.I. vectors} \end{array}\right\} \Rightarrow n \le m \end{array}\right\} \Rightarrow n = m$$

Definition Dimension

The dimension of a vector space V is defined to be the number of vectors in a basis for V

V: a vector space S: a basis for V

$\Rightarrow \dim(V) = \#(S)$ (the number of vectors in a basis S)

Definition Finite Dimensional Vector Space

- A vector space V is finite dimensional if it has a basis wioth a finite number of elements.
- If a vector space V is not finite dimensional, then it is called infinite dimensional.

Notes:

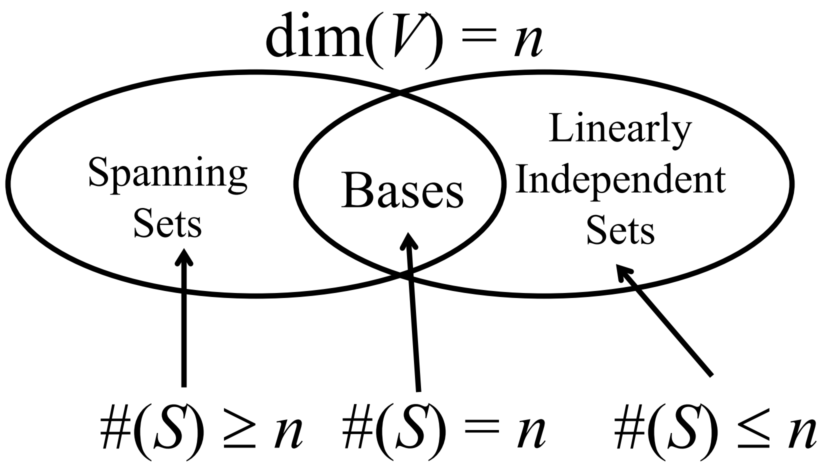
(1) $\dim(\{\mathbf{0}\}) = 0$ (If V consists of the zero vector alone, the dimension of V is defined as zero)

(2) Given $\dim(V) = n$, for $S \subseteq V$

S: a spanning set $\Rightarrow \#(S) \geq n$

S: a L.I. set $\Rightarrow \#(S) \leq n$

S: a basis $\Rightarrow \#(S) = n$



(3) Given $\dim(V) = n$, if W is a subspace of $V \Rightarrow \dim(W) \leq n$

Example Finding The Dimension From Standard Basis

(1) Vector space R^n $\Rightarrow$ standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ standard basis $\{E_{ij} \mid 1\le i\le m, 1\le j\le n\}$

and in E_{ij} $\begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space P_n $\Rightarrow$ standard basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n) = n+1$

(4) Vector space P_∞ $\Rightarrow$ standard basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P_\infty) = \infty$

Example Determining The Dimension of a Subspace

Find $\dim W$ for each of the following:

(a) $W = \{(d, c - d, c): c \text{ and } d \text{ are real numbers}\}$

(b) $W = \{(2b, b, 0): b \text{ is a real number}\}$

Solution

(a) $(d, c - d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

Example Determining The Dimension of a Subspace

Let W be the subspace of all symmetric matrices in $M_{2\times 2}$. What is the dimension of W?

Solution

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

$\Rightarrow S$ is a basis for W $\quad \Rightarrow \dim(W) = \#(S) = 3$

Exercise:

Which of the following is a basis for the subspace of $M_{2\times 2}$ of diagonal matrices?

A) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right\}$

B) $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\right\}$

C) $\left\{\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right\}$

D) $\left\{\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

Multiple Choice

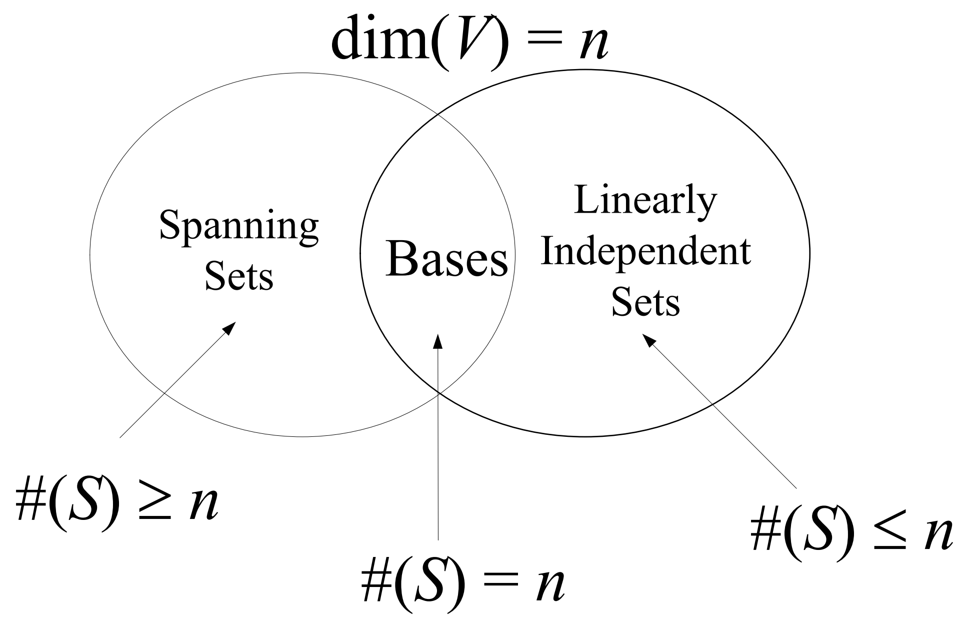
Theorem Methods to identify a basis in an n-dimensional space

Let V be a vector space of dimension n

(1) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V

(2) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ spans V, then S is a basis for V

(Both results are due to the fact that $\#(S) = n$)



Definition Coordinate Representation Relative to a Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

※ The "ordered" basis means the sequence of the vectors in the basis is specified

The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\mathbf{x}$ relative to the basis B. The coordinate matrix of $\mathbf{x}$ relative to B is a real-number column matrix whose components are the coordinates of $\mathbf{x}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example Coordinates in R^n

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3 relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Solution

$$\because \mathbf{x} = (-2,\ 1,\ 3) = -2(1,\ 0,\ 0) + 1(0,\ 1,\ 0) + 3(0,\ 0,\ 1)$$

$$\therefore [\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Example Finding the Coordinates relative to nonstandard basis in R^n

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3 relative to the following (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

Solution

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[\mathbf{x}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- Change of basis problem: You were given the coordinates of a vector relative to one basis B' and were asked to find the coordinates relative to another basis B

$$B = \{\mathbf{u}_1, \mathbf{u}_2\},\ B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$$ (B' is the original basis and B is the target basis)

If $[\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix},\ [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix}$ i.e., $\mathbf{u}'_1 = a\mathbf{u}_1 + b\mathbf{u}_2,\quad \mathbf{u}'_2 = c\mathbf{u}_1 + d\mathbf{u}_2$

Consider any $\mathbf{v} \in V,\ [\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$\begin{aligned} \Rightarrow \mathbf{v} &= k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 \\ &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2 \end{aligned}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$= \big[[\mathbf{u}'_1]_B\ [\mathbf{u}'_2]_B\big]\ [\mathbf{v}]_{B'} = {}_BP_{B'}\ [\mathbf{v}]_{B'}$$

In general, if $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathrm{B}' = \{\mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_n'\}$

${}_B P_{B'} = [[\mathbf{u}_1']_B \ [\mathbf{u}_2']_B \ldots [\mathbf{u}_n']_B]$ the transition matrix from B' to B

${}_{B'} P_B = \left[[\mathbf{u}_1]_{B'} \ [\mathbf{u}_2]_{B'} \ldots [\mathbf{u}_n]_{B'}\right]$ the transition matrix from B to B'

$$[\mathbf{v}]_B = {}_B P_{B'} \, [\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = {}_{B'} P_B \, [\mathbf{v}]_B$$

Theorem The inverse of a transition matrix

If ${}_{B}P_{B'}$ is the transition matrix from a basis B' to a basis B in R^n, then

(1) ${}_{B}P_{B'}$ is invertible

(2) The transition matrix ${}_{B'}P_{B}$ from B to B' is ${}_{\mathrm{B}}P_{B'}{}^{-1}$

Proof $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathrm{B}' = \{\mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_n'\}$

$$[\mathbf{v}]_B = [[\mathbf{u}_1']_B\ [\mathbf{u}_2']_B \ldots [\mathbf{u}_n']_B]\ [\mathbf{v}]_{B'} = {}_{B}P_{B'}\ [\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = \left[[\mathbf{u}_1]_{B'}\ [\mathbf{u}_2]_{B'} \ldots [\mathbf{u}_n]_{B'}\right]\ [\mathbf{v}]_B = {}_{B'}P_B\ [\mathbf{v}]_B$$

Replacing $[\mathbf{v}]_{B'}$ in the first equation with the second equation

$\Rightarrow [\mathbf{v}]_B = {}_{B}P_{B'}\,{}_{B'}P_B[\mathbf{v}]_B,$ for all $v \Rightarrow {}_{B}P_{B'}\,{}_{B'}P_B = I$

$\Rightarrow {}_{B}P_{B'}$ is invertible and ${}_{B}P_{B'}{}^{-1} = {}_{B'}P_B = \left[[\mathbf{u}_1]_{B'}\ [\mathbf{u}_2]_{B'} \ldots [\mathbf{u}_n]_{B'}\right]$

$\Rightarrow {}_{B}P_{B'}{}^{-1}$ is the transition matrix from B to B'

Theorem Deriving the transition matrix by G.-J. E.

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$ be two bases for R^n. Then the transition matrix ${}_B P_{B'}$ from B' to B can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B \vdots B']$ as follows

Construct the matrices *B* and *B'* by using ordered basis vectors as column vectors

Note that the target basis is always on the left

$$[B \vdots B'] \xrightarrow{\text{G.-J. E.}} [I_n \vdots {}_B P_{B'}]$$

Similarly, the transition matrix ${}_{B'} P_B$ from B to B' can be found via

$$[B' \vdots B] \xrightarrow{\text{G.-J. E.}} [I_n \vdots {}_{B'} P_B]$$

Example Finding a Transition Matrix

$B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B

(b) Let $[\mathbf{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\mathbf{v}]_B$

(c) Find the transition matrix from B to B'

Solution (a) For the original basis: $B' = \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}$. For the target basis: $B = \begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}$

$$\underbrace{\left[\begin{array}{cc|cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right]}_{B \qquad B'} \xrightarrow{\text{G.-J. E.}} \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]}_{I \qquad {}_B P_{B'}}$$

$$\therefore\ {}_B P_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$ (the transition matrix from B' to B)

(b)

$$[\mathbf{v}]_B = {}_B P_{B'} [\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(c)

$$\left[\begin{array}{cc:cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$\quad B' \qquad\qquad B \qquad\qquad\qquad\qquad I \qquad\qquad {}_{B'}P_B$$

$$\therefore\ {}_{B'}P_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$ (the transition matrix from B to B')

Check:

$${}_B P_{B'}\, {}_{B'}P_B = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example Coordinate Representation in P_3

Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the nonstandard basis in P_3, $S = \{1,\ 1 + x,\ 1 + x^2,\ 1 + x^3\}$

Solution

Solve $p = a(1) + b(1+x) + c(1+x^2) + d(1+x^3)$

$$\Rightarrow p = 3(1) + 0(1+x) + (-2)(1+x^2) + 3(1+x^3)$$

$$[p]_S = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

Example Coordinate Representation in $M_{2\times 2}$

Find the coordinate matrix of $x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ relative to the standard basis in M_{2x2}

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution

$$x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [x]_B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$