

4 Vector Spaces

4.3 Linear Independence

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4.3 Linear Independence

Definition Linear Independence (L.I.) and Linear Dependence (L.D.)

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$: a set of vectors in a vector space V

For $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$

- (1) If the equation has only the trivial solution ($c_1 = c_2 = \dots = c_k = 0$) then S (or $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$) is called linearly independent.
- (2) If the equation has a nontrivial solution (i.e., not all c_i 's are zero), then S (or $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$) is called linearly dependent.

Example

Testing For Linear Independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Solution

$$\begin{aligned} c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} &\Rightarrow \begin{aligned} c_1 - 2c_3 &= 0 \\ 2c_1 + c_2 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned} \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array} \right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

(or $\det(A) = -1 \neq 0$, so there is only the trivial solution)

Example**Testing For Linear Independence**

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1+x-2x^2, 2+5x-x^2, x+x^2\}$$

Solution

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$\text{i.e., } c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0+0x+0x^2$$

$$\Rightarrow \begin{array}{rcl} c_1 + 2c_2 & = & 0 \\ c_1 + 5c_2 + c_3 & = & 0 \\ -2c_1 - c_2 + c_3 & = & 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array} \right] \xrightarrow{\text{G.E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow$ This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2, c_2=-1, c_3=3$)

$\Rightarrow S$ is (or $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are) linearly dependent

Example

Testing For Linear Independence

Determine whether the following set of vectors in $M_{2 \times 2}$ is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

Solution

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \quad \Rightarrow \quad c_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcl} 2c_1 + 3c_2 + c_3 & = & 0 \\ c_1 & = & 0 \\ 2c_2 + 2c_3 & = & 0 \\ c_1 + c_2 & = & 0 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution) $\Rightarrow S$ is linearly independent

Theorem

A Defining Property of L.D.

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, for $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ in S can be written as a linear combination of the other vectors in S .

Proof. $(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_i\mathbf{v}_i + \dots + c_k\mathbf{v}_k = \mathbf{0}$

$\because S$ is linearly dependent (there exist nontrivial solutions) $\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \left(-\frac{c_1}{c_i}\right)\mathbf{v}_1 + \dots + \left(-\frac{c_{i-1}}{c_i}\right)\mathbf{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right)\mathbf{v}_{i+1} + \dots + \left(-\frac{c_k}{c_i}\right)\mathbf{v}_k$$

$(\Leftarrow)$ Let $\mathbf{v}_i = d_1\mathbf{v}_1 + \dots + d_{i-1}\mathbf{v}_{i-1} + d_{i+1}\mathbf{v}_{i+1} + \dots + d_k\mathbf{v}_k$

$$\Rightarrow d_1\mathbf{v}_1 + \dots + d_{i-1}\mathbf{v}_{i-1} + (-1)\mathbf{v}_i + d_{i+1}\mathbf{v}_{i+1} + \dots + d_k\mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = d_1, c_2 = d_2, \dots, c_i = -1, \dots, c_k = d_k \quad (\text{there exists at least this nontrivial solution})$$

$\Rightarrow S$ is linearly dependent

Theorem

Easily Checked Sets

- (a) A finite set that contains 0 is linearly dependent.
- (b) A set with exactly one vector is linearly independent if and only if that vector is not 0.
- (c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Proof.

- (a) $S = \{0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is L.D. since $1 \cdot 0 + 0v_1 + 0v_2 + \dots + 0v_k = 0$.
- (b) Follows from the fact that $cv = 0 \iff c = 0 \text{ or } v = 0$.
- (c) Follows from the previous theorem and is left as an exercise.

Theorem

Checking L.D. From Number of Vectors

Let $S = \{v_1, v_2, \dots, v_r\} \subseteq R^n$. If $r > n$, then S is L.D..

Proof The vector equation $c_1v_1 + c_2v_2 + \dots + c_rv_r = 0$ gives a homogeneous linear System of n equations in r variables. Since $r > n$, it must have a nontrivial solution. So, the set S must be L.D..

4.4 Coordinates and Basis

4.5 Dimension

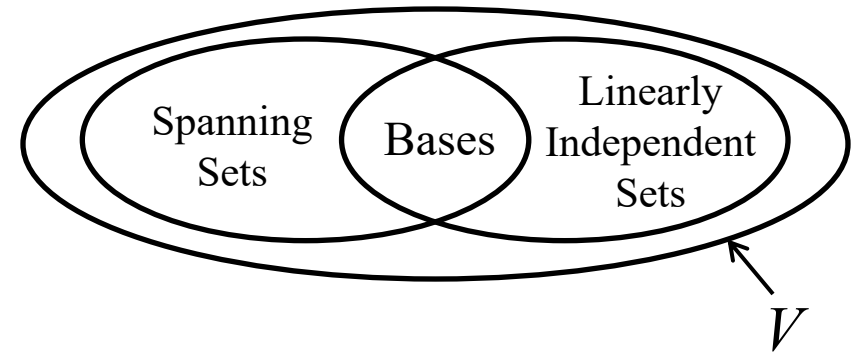
4.6 Change of Basis

Definition

Basis

$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$: a set of vectors in a vector space V is called a basis if

- (1) S is linearly independent, and
- (2) S spans V , i.e., $\text{span}(S) = V$.



Note: A basis S must have enough vectors to span V , but not so many vectors that one of them could be written as a linear combination of the other vectors in S .

Example

A (Standard) Basis For R^3

$S = \{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for R^3 :

- (1) S is L.I. for if $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$, then $c_1 = c_2 = c_3 = 0$.
- (2) $\text{span}(S) = V$, since $(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$.

Example**A (Standard) Basis For R^n**

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}, \text{ where } \mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 1)$$

Example**A (Standard) Basis For $M_{m \times n}$**

$$S = \{ E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \}, \text{ and in } E_{ij} \begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$$

E.g. For $M_{2 \times 2}$ we have:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Example

A (Standard) Basis For P_n

$$S = \{1, x, x^2, \dots, x^n\} \quad \text{E.g. for } P_3(x): \quad S = \{1, x, x^2, x^3\}$$

Example

A Nonstandard Basis For R^2

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 1), (1, -1)\}$ is a basis for R^2

Solution

(1) For any $\mathbf{u} = (u_1, u_2) \in R^2$, $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{u} \Rightarrow \begin{cases} c_1 + c_2 = u_1 \\ c_1 - c_2 = u_2 \end{cases}$ Because the coefficient matrix of this system has a **nonzero determinant**, the system has a unique solution for each $\mathbf{u}$. Thus S spans R^2

(2) For $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases}$ Because the coefficient matrix of this system has a **nonzero determinant**, we know that the system has only the trivial solution. Thus S is linearly independent

From (1) and (2) we conclude that S is a (nonstandard) basis for R^2

Theorem

Uniqueness of Basis Representation of Vectors

Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V , then every vector v in V can be written in one and only one way as a linear combination of vectors in S .

Proof

$$\because S \text{ is a basis} \Rightarrow \begin{cases} (1) \text{ span}(S) = V \\ (2) S \text{ is linearly independent} \end{cases}$$

$$\because \text{span}(S) = V \quad \text{Let } \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

$$\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n$$

$$\Rightarrow \mathbf{v} + (-1)\mathbf{v} = \mathbf{0} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \dots + (c_n - b_n)\mathbf{v}_n$$

$$\because S \text{ is linearly independent} \Rightarrow \text{with only the trivial solution}$$

$$\Rightarrow \text{coefficients for } \mathbf{v}_i \text{ are all zero}$$

$$\Rightarrow c_1 = b_1, c_2 = b_2, \dots, c_n = b_n \text{ (i.e., unique basis representation)}$$

Theorem

Uniqueness of Basis Representation of Vectors

Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V , then every set containing more than n vectors in V is linearly dependent. (In other words, every linearly independent set contains at most n vectors)

Proof

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$, $m > n$

$\because \text{span}(S) = V$

$$\begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \dots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_i \in V \quad \Rightarrow \quad \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \dots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \dots + c_{nm}\mathbf{v}_n \end{aligned}$$

Consider $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\dots+k_m\mathbf{u}_m=\mathbf{0}$

(if k_i 's are not all zero, S_1 is linearly dependent)

$$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\dots+d_n\mathbf{v}_n=\mathbf{0} \quad (d_i=c_{i1}k_1+c_{i2}k_2+\dots+c_{im}k_m)$$

$$\begin{aligned} \because S \text{ is L.I. } \Rightarrow d_i=0 \quad \forall i \quad \text{i.e., } & c_{11}k_1+c_{12}k_2+\dots+c_{1m}k_m=0 \\ & c_{21}k_1+c_{22}k_2+\dots+c_{2m}k_m=0 \\ & \vdots \\ & c_{n1}k_1+c_{n2}k_2+\dots+c_{nm}k_m=0 \end{aligned}$$

$\because$ This is a homogeneous system with more variables ($k_1, k_2, \dots, k_m$), than equations (n equations), then it must have infinitely many solutions

$\Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\dots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial (nonzero) solution $\Rightarrow S_1$ is L.D.

Note: This theorem tells us that if V has a bas with n vectors then every independent set must have at most n vectors.

Theorem

Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Proof Using the note in the previous slide:

$$\begin{aligned} S &= \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \\ S' &= \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\} \end{aligned} \text{ are two bases for } V$$

$$\left. \begin{array}{l} S \text{ is a basis } V \\ S' \text{ is a set of L.I. vectors} \end{array} \right\} \Rightarrow m \leq n$$
$$\left. \begin{array}{l} S' \text{ is a basis } V \\ S \text{ is a set of L.I. vectors} \end{array} \right\} \Rightarrow n \leq m$$
$$\left. \begin{array}{l} \Rightarrow m \leq n \\ \Rightarrow n \leq m \end{array} \right\} \Rightarrow n = m$$

Definition

Dimension

The dimension of a vector space V is defined to be the number of vectors in a basis for V

V : a vector space S : a basis for V

$$\Rightarrow \dim(V) = \#(S) \quad (\text{the number of vectors in a basis } S)$$

Definition

Finite Dimensional Vector Space

- A vector space V is finite dimensional if it has a basis with a finite number of elements.
- If a vector space V is not finite dimensional, then it is called infinite dimensional.

Notes:

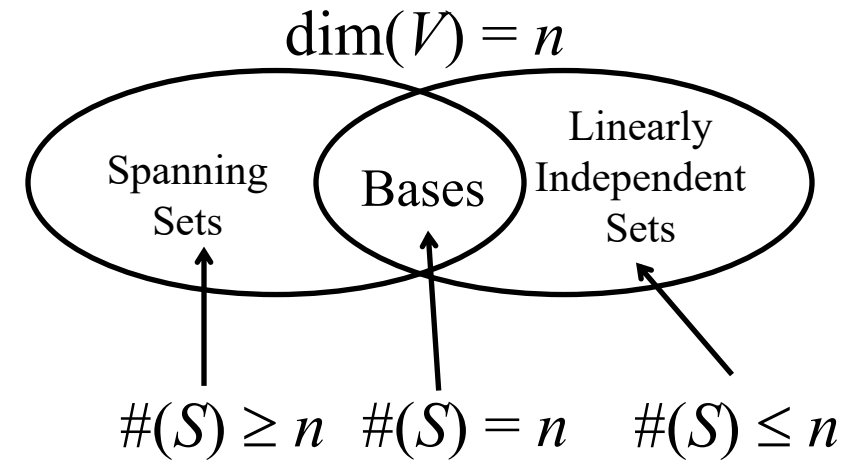
(1) $\dim(\{\mathbf{0}\}) = 0$ (If V consists of the zero vector alone, the dimension of V is defined as zero)

(2) Given $\dim(V) = n$, for $S \subseteq V$

S : a spanning set $\Rightarrow \#(S) \geq n$

S : a L.I. set $\Rightarrow \#(S) \leq n$

S : a basis $\Rightarrow \#(S) = n$



(3) Given $\dim(V) = n$, if W is a subspace of $V \Rightarrow \dim(W) \leq n$

Example

Finding The Dimension From Standard Basis

(1) Vector space $R^n \Rightarrow$ standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$

$$\Rightarrow \dim(R^n) = n$$

(2) Vector space $M_{m \times n} \Rightarrow$ standard basis $\{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$

$$\text{and in } E_{ij} \begin{cases} a_{ij} = 1 \\ \text{other entries are zero} \end{cases}$$

$$\Rightarrow \dim(M_{m \times n}) = mn$$

(3) Vector space $P_n \Rightarrow$ standard basis $\{1, x, x^2, \dots, x^n\}$

$$\Rightarrow \dim(P_n) = n+1$$

(4) Vector space $P_\infty \Rightarrow$ standard basis $\{1, x, x^2, \dots\}$

$$\Rightarrow \dim(P_\infty) = \infty$$

Example**Determining The Dimension of a Subspace**

Find $\dim W$ for each of the following:

(a) $W = \{(d, c - d, c) : c \text{ and } d \text{ are real numbers}\}$

(b) $W = \{(2b, b, 0) : b \text{ is a real number}\}$

Solution

(a) $(d, c - d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

Example**Determining The Dimension of a Subspace**

Let W be the subspace of all symmetric matrices in $M_{2 \times 2}$. What is the dimension of W ?

Solution

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

$$\Rightarrow S \text{ is a basis for } W \Rightarrow \dim(W) = \#(S) = 3$$

Theorem

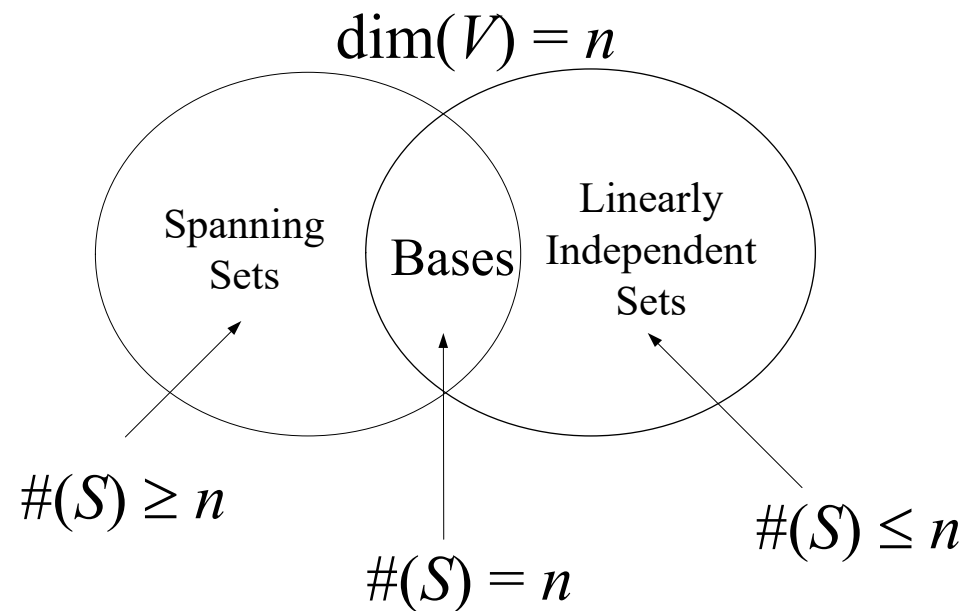
Methods to identify a basis in an n -dimensional space

Let V be a vector space of dimension n

(1) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V , then S is a basis for V

(2) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V , then S is a basis for V

(Both results are due to the fact that $\#(S) = n$)



Definition

Coordinate Representation Relative to a Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n.$$

※ The “ordered” basis means the sequence of the vectors in the basis is specified

The scalars $c_1, c_2, \dots, c_n$ are called the **coordinates of $\mathbf{x}$ relative to the basis B** . The coordinate matrix of $\mathbf{x}$ relative to B is a real-number column matrix whose components are the coordinates of $\mathbf{x}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Example**Coordinates in R^n**

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3 relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Solution

$$\because \mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$\therefore [\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Example Finding the Coordinates relative to nonstandard basis in R^n

Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3 relative to the following (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

Solution

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$

$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\begin{array}{rrcr} c_1 & & + & 2c_3 & = & 1 \\ & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[\mathbf{x}]_{B'} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- **Change of basis problem:** You were given the coordinates of a vector relative to one basis B' and were asked to find the coordinates relative to another basis B

$$B = \{\mathbf{u}_1, \mathbf{u}_2\}, B' = \{\mathbf{u}'_1, \mathbf{u}'_2\} \quad (B' \text{ is the original basis and } B \text{ is the target basis})$$

$$\text{If } [\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix}, [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix} \quad \text{i.e., } \mathbf{u}'_1 = a\mathbf{u}_1 + b\mathbf{u}_2, \quad \mathbf{u}'_2 = c\mathbf{u}_1 + d\mathbf{u}_2$$

$$\begin{aligned} \text{Consider any } \mathbf{v} \in V, [\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &\Rightarrow \mathbf{v} = k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 \\ &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2 \\ &\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \begin{bmatrix} [\mathbf{u}'_1]_B & [\mathbf{u}'_2]_B \end{bmatrix} [\mathbf{v}]_{B'} = {}_B P_{B'} [\mathbf{v}]_{B'} \end{aligned}$$

In general, if $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$

${}_B P_{B'} = [[\mathbf{u}'_1]_B \ [\mathbf{u}'_2]_B \ \dots \ [\mathbf{u}'_n]_B]$ the transition matrix from B' to B

${}_{B'} P_B = [[\mathbf{u}_1]_{B'} \ [\mathbf{u}_2]_{B'} \ \dots \ [\mathbf{u}_n]_{B'}]$ the transition matrix from B to B'

$$[\mathbf{v}]_B = {}_B P_{B'} [\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = {}_{B'} P_B [\mathbf{v}]_B$$

Theorem

The inverse of a transition matrix

If ${}_B P_{B'}$ is the transition matrix from a basis B' to a basis B in R^n , then

(1) ${}_B P_{B'}$ is invertible

(2) The transition matrix ${}_B P_{B'}$ from B to B' is ${}_B P_{B'}^{-1}$

Proof $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$

$$[\mathbf{v}]_B = [[\mathbf{u}'_1]_B \ [\mathbf{u}'_2]_B \ \dots \ [\mathbf{u}'_n]_B] [\mathbf{v}]_{B'} = {}_B P_{B'} [\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = [[\mathbf{u}_1]_{B'} \ [\mathbf{u}_2]_{B'} \ \dots \ [\mathbf{u}_n]_{B'}] [\mathbf{v}]_B = {}_{B'} P_B [\mathbf{v}]_B$$

Replacing $[\mathbf{v}]_{B'}$ in the first equation with the second equation

$$\Rightarrow [\mathbf{v}]_B = {}_B P_{B'} {}_{B'} P_B [\mathbf{v}]_B, \text{ for all } v \Rightarrow {}_B P_{B'} {}_{B'} P_B = I$$

$$\Rightarrow {}_B P_{B'} \text{ is invertible and } {}_B P_{B'}^{-1} = {}_{B'} P_B = [[\mathbf{u}_1]_{B'} \ [\mathbf{u}_2]_{B'} \ \dots \ [\mathbf{u}_n]_{B'}]$$

$$\Rightarrow {}_B P_{B'}^{-1} \text{ is the transition matrix from } B \text{ to } B'$$

Theorem

Deriving the transition matrix by G.-J. E.

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_n\}$ be two bases for R^n . Then the **transition matrix** ${}_B P_{B'}$ from B' to B can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B:B']$ as follows

Construct the matrices B and B' by using ordered basis vectors as column vectors

Note that the target basis is always on the left

$$[B:B'] \xrightarrow{\text{G.-J. E.}} [I_n \vdots {}_B P_{B'}]$$

Similarly, the **transition matrix** ${}_B P_B$ from B to B' can be found via

$$[B':B] \xrightarrow{\text{G.-J. E.}} [I_n \vdots {}_{B'} P_B]$$

Example**Finding a Transition Matrix**

$B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ are two bases for $\mathbb{R}^2$

(a) Find the transition matrix from B' to B

(b) Let $[\mathbf{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\mathbf{v}]_B$

(c) Find the transition matrix from B to B'

For the original basis: $B' = \begin{bmatrix} -1 & 2 \\ 2 & -2 \end{bmatrix}$

For the target basis: $B = \begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}$

Solution

(a)

$$\begin{array}{ccc} \begin{bmatrix} -3 & 4 & \vdots & -1 & 2 \\ 2 & -2 & \vdots & 2 & -2 \end{bmatrix} & \xrightarrow{\text{G.-J. E.}} & \begin{bmatrix} 1 & 0 & \vdots & 3 & -2 \\ 0 & 1 & \vdots & 2 & -1 \end{bmatrix} \\ B & B' & I \quad {}_B P_{B'} \end{array}$$

$$\therefore {}_B P_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ (the transition matrix from } B' \text{ to } B)$$

(b)

$$[\mathbf{v}]_B = {}_B P_{B'} [\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(c)

$$\begin{array}{ccc} \left[\begin{array}{cc|cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array} \right] & \xrightarrow{\text{G.-J. E.}} & \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array} \right] \\ B' & B & I \quad {}_B P_{B'} \end{array}$$

$$\therefore {}_{B'} P_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad (\text{the transition matrix from } B \text{ to } B')$$

Check:

$${}_B P_{{}_{B'} B'} P_B = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example**Coordinate Representation in P_3**

Find the coordinate matrix of $p = 3x^3 - 2x^2 + 4$ relative to the nonstandard basis in P_3 , $S = \{1, 1 + x, 1 + x^2, 1 + x^3\}$

Solution

$$\text{Solve } p = a(1) + b(1+x) + c(1+x^2) + d(1+x^3)$$

$$\Rightarrow p = 3(1) + 0(1+x) + (-2)(1+x^2) + 3(1+x^3)$$

$$[p]_S = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

Example**Coordinate Representation in $M_{2 \times 2}$**

Find the coordinate matrix of $x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ relative to the standard basis in $M_{2 \times 2}$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution

$$x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 5 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [x]_B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$