UNIT 3B

# Independence, Basis and Dimension

Unit 3B · Sections 4.3 – 4.6

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

# What you should be able to do

## Sections 4.3 and 4.4

- Decide whether vectors are linearly independent by solving $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$, and recognise the easy cases: sets containing $\mathbf{0}$, sets of one or two vectors, more than $n$ vectors in $\mathbb{R}^n$.
- Check that a set is a basis and find the coordinate vector of a vector relative to a basis.

## Sections 4.5 and 4.6

- Find the dimension of a space, of a span and of the solution space of $A\mathbf{x} = \mathbf{0}$; use the dimension to shorten basis checks.
- Reduce a spanning set, or enlarge a linearly independent set, to a basis.
- Find transition matrices between two bases and use them to convert coordinates.

SECTION 4.3

# Linear Independence

**GOAL** Decide whether a set of vectors contains a redundant vector, one that is a linear combination of the others. A set without one reaches every vector of its span in exactly one way.

1. One axis too many
2. Linearly independent sets, and the test for them
3. Examples in $\mathbb{R}^n$, $P_n$ and $M_{22}$
4. Sets that can be checked at a glance
5. Pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$
6. More than $n$ vectors in $\mathbb{R}^n$

# One axis too many

In the $xy$-plane, $(3,2)$ can be written in only one way with $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$:

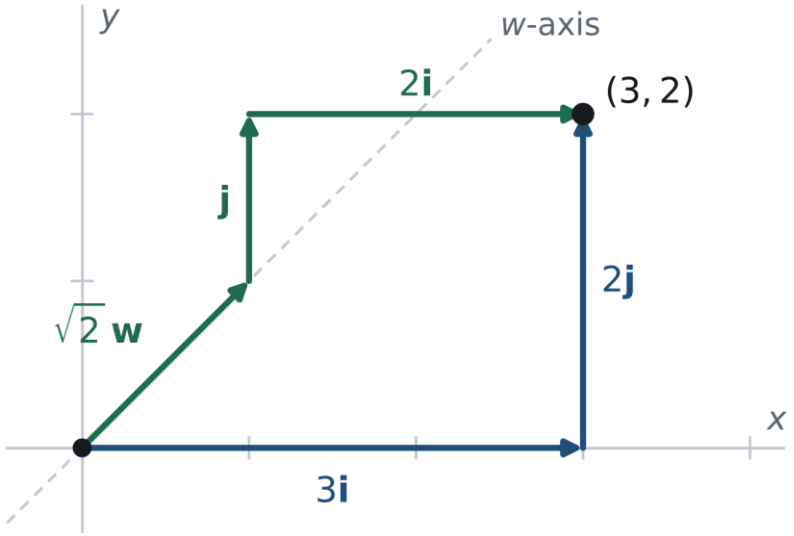
$$(3,2) = 3\mathbf{i} + 2\mathbf{j}$$

Now add a third axis at $45°$, the $w$-axis, with unit vector

$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Then $(3,2)$ can be written in infinitely many ways, for example

$$(3,2) = 3\mathbf{i} + 2\mathbf{j} + 0\mathbf{w} = 2\mathbf{i} + \mathbf{j} + \sqrt{2}\,\mathbf{w} = 4\mathbf{i} + 3\mathbf{j} - \sqrt{2}\,\mathbf{w}$$




> The third axis is SUPERFLUOUS: $\mathbf{w} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ is a linear combination of $\mathbf{i}$ and $\mathbf{j}$, so it adds no new direction, only new ways of writing the same vectors. A set with no such redundant vector is called LINEARLY INDEPENDENT.

DEFINITION 1

# Linearly independent and linearly dependent sets

**DEFINITION 1** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ be a set of two or more vectors in a vector space $V$. $S$ is **LINEARLY INDEPENDENT** if no vector in $S$ is a linear combination of the others. Otherwise $S$ is **LINEARLY DEPENDENT**. A set $\{\mathbf{v}\}$ with one vector is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.

Linearly dependent means that $S$ has a REDUNDANT vector: one that the others already produce. For example:

- $\{(1,0),\ (0,1),\ (1,1)\}$ is linearly dependent, since $(1,1) = (1,0) + (0,1)$.
- $\{(1,2,3),\ (2,4,6)\}$ is linearly dependent, since $(2,4,6) = 2(1,2,3)$.
- $\{(1,0),\ (0,1)\}$ is linearly independent: neither vector is a multiple of the other.

Abbreviations in the exercises: L.I. = linearly independent, L.D. = linearly dependent.

THEOREM 4.3.1

# The test for linear independence

> **THEOREM 4.3.1** A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space $V$ is linearly independent if and only if the ONLY scalars with $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ are $k_1 = k_2 = \cdots = k_r = 0$.

## How to use it

Step 1. Write $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$.

Step 2. Equate components (or coefficients, or entries): this gives a HOMOGENEOUS linear system in $k_1, \ldots, k_r$.

Step 3. Only the trivial solution: linearly INDEPENDENT. A nontrivial solution as well: linearly DEPENDENT.

- Example 1: $k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n = (k_1, k_2, \ldots, k_n)$, which is $\mathbf{0}$ only when every $k_i = 0$. So the standard unit vectors of $\mathbb{R}^n$ are linearly independent.

Shortcut for $n$ vectors in $\mathbb{R}^n$: they are linearly independent exactly when $\det[\,\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n\,] \neq 0$ (Theorem 2.3.8).

PROOF

# Why the test works

For one vector: $k\mathbf{v} = \mathbf{0}$ forces $k = 0$ exactly when $\mathbf{v} \neq \mathbf{0}$ (Theorem 4.1.1(d)). Now let $r \geq 2$.

**A vector is a combination of the others $\Rightarrow$ there is a nontrivial solution**

Number the vectors so that $\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r$. Then

$$(-1)\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = \mathbf{0},$$

and the coefficient $-1$ is not zero.

**There is a nontrivial solution $\Rightarrow$ a vector is a combination of the others**

Number the vectors so that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ with $k_1 \neq 0$. Divide by $k_1$:

$$\mathbf{v}_1 = \left(-\frac{k_2}{k_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\mathbf{v}_r$$

So $S$ is linearly dependent if and only if a nontrivial solution exists, which is the theorem. ■

EXAMPLE 2

# Three vectors in $\mathbb{R}^3$

Are $\mathbf{v}_1 = (1,-2,3)$, $\mathbf{v}_2 = (5,6,-1)$, $\mathbf{v}_3 = (3,2,1)$ linearly independent? Equate components in $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$:

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

The columns of the coefficient matrix are $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 5 & 3 & | & 0 \\ -2 & 6 & 2 & | & 0 \\ 3 & -1 & 1 & | & 0 \end{bmatrix} \xrightarrow{A_{1,2}^{(2)},\, A_{1,3}^{(-3)}} \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 0 & 16 & 8 & | & 0 \\ 0 & -16 & -8 & | & 0 \end{bmatrix} \xrightarrow{A_{2,3}^{(1)},\, M_2^{(1/8)}} \begin{bmatrix} 1 & 5 & 3 & | & 0 \\ 0 & 2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$k_3$ is free. With $k_3 = 2t$: $k_2 = -t$ and $k_1 = -5k_2 - 3k_3 = -t$. NONTRIVIAL solutions exist, so the vectors are linearly DEPENDENT. Taking $t = 1$:

$$-\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}, \qquad \text{so} \qquad \mathbf{v}_3 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$$

Quicker here: three vectors in $\mathbb{R}^3$, and $\det[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\,] = 0$.

# No determinant available: four equations, three unknowns

### Example 3: three vectors in $\mathbb{R}^4$

$\mathbf{v}_1 = (1,2,2,-1)$, $\mathbf{v}_2 = (4,9,9,-4)$, $\mathbf{v}_3 = (5,8,9,-5)$. The system $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ has the coefficient matrix on the left:

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 9 & 8 \\ 2 & 9 & 9 \\ -1 & -4 & -5 \end{bmatrix} \xrightarrow{A_{1,2}^{(-2)},\ A_{1,3}^{(-2)},\ A_{1,4}^{(1)},\ A_{2,3}^{(-1)}} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A pivot in every column: no free variable, so ONLY the trivial solution. Linearly INDEPENDENT.

### Three matrices in $M_{22}$

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}.$$

$c_1A_1 + c_2A_2 + c_3A_3 = O$; equate the four entries:

$$\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}$$

The second equation gives $c_1 = 0$, then the fourth $c_2 = 0$, then the third $c_3 = 0$. Only the trivial solution: linearly INDEPENDENT.

# Polynomials

**Example 4:** $\{1, x, x^2, \ldots, x^n\}$ **is linearly independent in** $P_n$

If $a_0 + a_1x + \cdots + a_nx^n = 0$ for EVERY real $x$, then all $a_i = 0$: a nonzero polynomial of degree at most $n$ has at most $n$ roots, not infinitely many.

**Example 5:** $\mathbf{p}_1 = 1 - x$, $\mathbf{p}_2 = 5 + 3x - 2x^2$, $\mathbf{p}_3 = 1 + 3x - x^2$ **in** $P_2$

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = (k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)\,x + (-2k_2 - k_3)\,x^2$$

This is the zero polynomial only if every coefficient is 0:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned}$$

The coefficient matrix has determinant 0, so there are nontrivial solutions, for example $k_1 = -3$, $k_2 = 1$, $k_3 = -2$. Linearly DEPENDENT:

$$\mathbf{p}_2 = 3\mathbf{p}_1 + 2\mathbf{p}_3$$

> The columns of the coefficient matrix, $(1, -1, 0)$, $(5, 3, -2)$ and $(1, 3, -1)$, are the coefficient lists of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$: a question in $P_2$ became a question in $\mathbb{R}^3$. Section 4.4 explains why (coordinate vectors).

THEOREM 4.3.2

# Sets that can be checked at a glance

**THEOREM 4.3.2** In a vector space $V$:

**(a)** A finite set that contains $\mathbf{0}$ is linearly dependent.

**(b)** A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

**(c)** A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

**Proof of (a)**

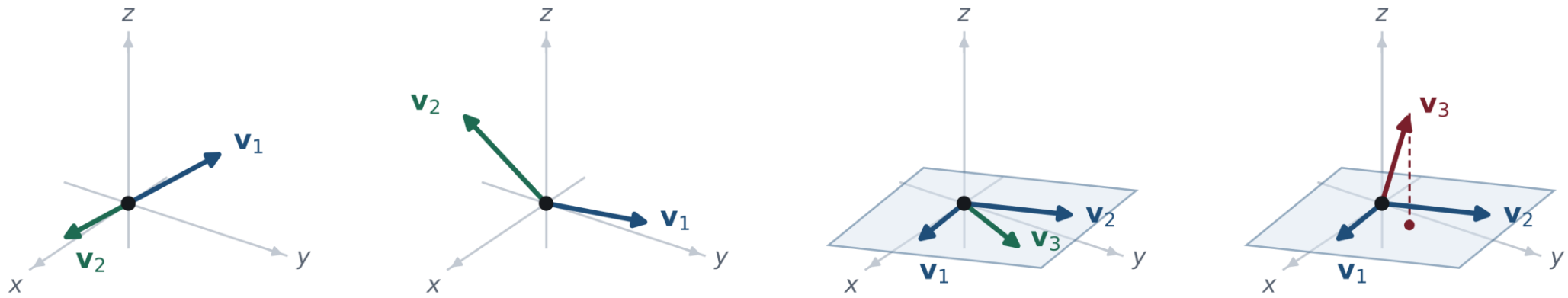
For any vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$: $0\mathbf{v}_1 + \cdots + 0\mathbf{v}_r + 1\,\mathbf{0} = \mathbf{0}$ is a nontrivial solution, because the coefficient of $\mathbf{0}$ is 1. ■

(b) is Theorem 4.1.1(d), and (c) is Definition 1 for two vectors.

**Example 6: two functions in $F(-\infty, \infty)$**

- $f_1 = x$ and $f_2 = \sin x$: neither is a multiple of the other, so $\{f_1, f_2\}$ is linearly independent.
- $g_1 = \sin 2x$ and $g_2 = \sin x \cos x$: since $\sin 2x = 2\sin x \cos x$, $g_1 = 2g_2$, so $\{g_1, g_2\}$ is linearly dependent.

# Linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$




(a) dependent: one line (b) independent (c) dependent: one plane (d) independent

- Two vectors, placed with their initial points at the origin, are linearly independent if and only if they do NOT lie on one line.
- Three vectors in $\mathbb{R}^3$, placed with their initial points at the origin, are linearly independent if and only if they do NOT lie in one plane.

> Dependent means that some vector adds no new direction: it lies on the line, or in the plane, of the others.

THEOREM 4.3.3

# Too many vectors

**THEOREM 4.3.3** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then $S$ is linearly dependent.

**Proof**

Equating components in $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ gives a homogeneous system of $n$ equations in $r$ unknowns. It has more unknowns than equations, so it has nontrivial solutions (Theorem 1.2.2). ■

Three or more vectors in $\mathbb{R}^2$, four or more in $\mathbb{R}^3$: always linearly dependent, e.g. $\{(3,-1),\ (4,5),\ (-4,7)\}$ in $\mathbb{R}^2$.

This is why the $w$-axis was superfluous: $\mathbf{i}, \mathbf{j}, \mathbf{w}$ are three vectors in $\mathbb{R}^2$.

The converse is false: $r \leq n$ does NOT make a set independent, e.g. $\{(1,2,3),\ (2,4,6)\}$ in $\mathbb{R}^3$.

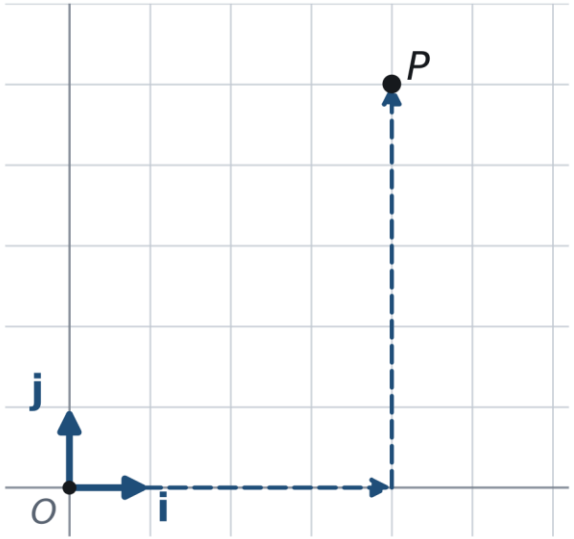
SECTION 4.4

# Coordinates and Basis

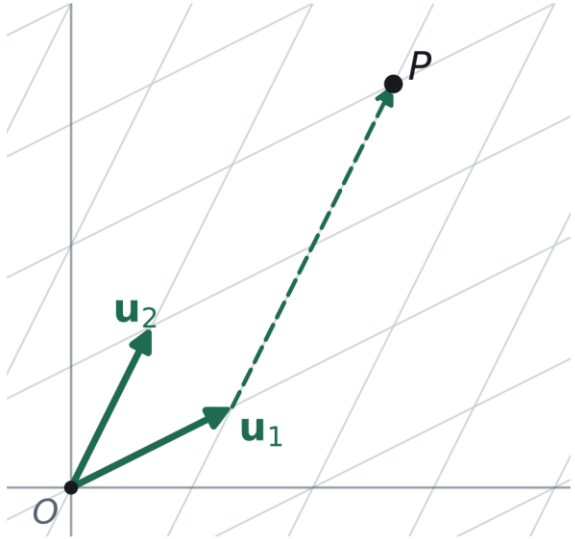
**GOAL** A basis is a coordinate system for a vector space: enough vectors to reach every vector (it spans), and none to spare (it is linearly independent). Then every vector has exactly one list of coordinates.

1. Coordinate systems with skewed axes
2. Basis: a spanning set that is linearly independent
3. Standard bases of $\mathbb{R}^n$, $P_n$ and $M_{mn}$; another basis of $\mathbb{R}^3$
4. Finite- and infinite-dimensional spaces
5. Unique representation, and the coordinate vector $(\mathbf{v})_S$

# Coordinates come from a choice of basis vectors




standard basis $\{\mathbf{i}, \mathbf{j}\}$: $P = 4\mathbf{i} + 5\mathbf{j}$, coordinates $(4, 5)$




basis $\{\mathbf{u}_1, \mathbf{u}_2\}$: $P = 1\mathbf{u}_1 + 2\mathbf{u}_2$, coordinates $(1, 2)$

Any two vectors of the plane that do NOT lie on one line can serve as axes: their directions fix the positive directions, and their lengths fix the spacing. The coordinates of a point $P$ are the coefficients in $\overrightarrow{OP} = a\mathbf{u}_1 + b\mathbf{u}_2$.

> Here $P = (4,5)$ in the usual coordinates, and $\overrightarrow{OP} = 1\,\mathbf{u}_1 + 2\,\mathbf{u}_2$ for $\mathbf{u}_1 = (2,1)$, $\mathbf{u}_2 = (1,2)$: the same point has coordinates $(1,2)$ in the second system.

DEFINITION 1

# Basis

**DEFINITION** A vector space $V$ is FINITE-DIMENSIONAL if some finite set of vectors spans $V$, and INFINITE-DIMENSIONAL if no finite set spans $V$.

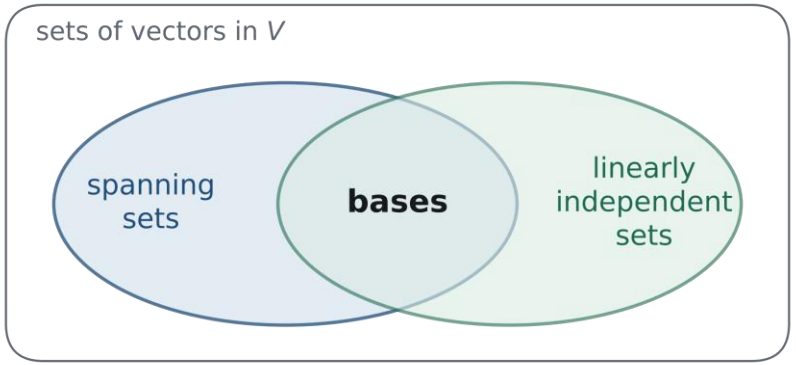
**DEFINITION 1** A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of vectors in a finite-dimensional vector space $V$ is a BASIS for $V$ if

**1)** $S$ spans $V$, and

**2)** $S$ is linearly independent.

- Condition 1) gives ENOUGH vectors: every vector of $V$ is a combination of them.
- Condition 2) gives NONE TO SPARE: no vector of $S$ is a combination of the others.

A basis has enough vectors to span $V$, but not so many that one of them is a linear combination of the others.



EXAMPLES 1, 2 AND 4

# The standard bases

**$\mathbb{R}^n$: $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, with $\mathbf{e}_1 = (1,0,\ldots,0)$, …, $\mathbf{e}_n = (0,\ldots,0,1)$**

They span $\mathbb{R}^n$ (Section 4.2) and are linearly independent (Section 4.3, Example 1). In $\mathbb{R}^3$ this is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

**$P_n$:** $\{1, x, x^2, \ldots, x^n\}$

They span $P_n$ (Section 4.2) and are linearly independent (Section 4.3, Example 4).

**$M_{22}$: the four matrix units**

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

This sum is $O$ only when every $c_i = 0$ (independent), and it equals any 2×2 matrix with entries $a, b, c, d$ for $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$ (spanning).

- In $M_{mn}$ the $mn$ matrices with one entry 1 and all other entries 0 form the standard basis.

The zero space $\{\mathbf{0}\}$: by definition, its basis is the empty set $\varnothing$.

EXAMPLE 3

# Another basis for $\mathbb{R}^3$

Show that $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (2,9,0)$, $\mathbf{v}_3 = (3,3,4)$ form a basis for $\mathbb{R}^3$.

- Independent: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution.
- Spanning: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$ must have a solution for every $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

Equating components, both are linear systems with the SAME coefficient matrix, whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}, \qquad \det(A) = 1(36-0) - 2(8-3) + 3(0-9) = -1 \neq 0$$

By Theorem 2.3.8, $\det(A) \neq 0$ settles BOTH questions: $A\mathbf{c} = \mathbf{0}$ has only the trivial solution, and $A\mathbf{c} = \mathbf{b}$ has a solution for every $\mathbf{b}$. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

> A vector space has MANY bases: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are both bases for $\mathbb{R}^3$. For $n$ vectors in $\mathbb{R}^n$: a basis exactly when $\det[\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_n\,] \neq 0$.

# Infinite-dimensional spaces

### $P_\infty$, all polynomials, has no finite spanning set

Suppose $S = \{\mathbf{p}_1, \dots, \mathbf{p}_r\}$ spanned $P_\infty$, and let $n$ be the largest degree among $\mathbf{p}_1, \dots, \mathbf{p}_r$. Every linear combination of them has degree at most $n$, so $x^{n+1}$ is not a linear combination of them, which is a contradiction.

- Finite-dimensional: $\mathbb{R}^n$, $P_n$, $M_{mn}$ and $\{\mathbf{0}\}$, which have finite bases.
- Infinite-dimensional: $P_\infty$, $\mathbb{R}^\infty$, $F(-\infty, \infty)$, $C(-\infty, \infty)$, $C^m(-\infty, \infty)$ and $C^\infty(-\infty, \infty)$.

> From here on the spaces are finite-dimensional: they have bases, and everything in Sections 4.4 to 4.6 applies to them.

THEOREM 4.4.1

# Every vector has exactly one representation

> **THEOREM 4.4.1** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector $\mathbf{v} \in V$ can be written as $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in EXACTLY ONE way.

**Proof**

At least one way, because $S$ spans $V$. At most one way: suppose that also $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$. Subtracting the two expressions,

$$\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

$S$ is linearly independent, so every coefficient is 0: $c_1 = k_1,\ c_2 = k_2,\ \ldots,\ c_n = k_n.$ ■

Spanning gives the EXISTENCE of the coefficients; independence gives their UNIQUENESS. The set $\{\mathbf{i}, \mathbf{j}, \mathbf{w}\}$ of Section 4.3 spans $\mathbb{R}^2$ but is dependent, and its coefficients were not unique.

DEFINITION 2

# Coordinates relative to a basis

**DEFINITION 2** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis for $V$ and $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. The scalars $c_1, c_2, \dots, c_n$ are the COORDINATES of $\mathbf{v}$ relative to $S$, and the vector $(\mathbf{v})_S = (c_1, c_2, \dots, c_n)$ of $\mathbb{R}^n$ is the COORDINATE VECTOR of $\mathbf{v}$ relative to $S$.

Written as a column, the coordinate vector is denoted $[\mathbf{v}]_S$ (its MATRIX FORM):

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- The ORDER of the basis vectors matters: listing $\mathbf{v}_2$ before $\mathbf{v}_1$ swaps $c_1$ and $c_2$, and $(1,2) \neq (2,1)$. A basis used for coordinates is an ORDERED basis.

By Theorem 4.4.1, $\mathbf{v} \mapsto (\mathbf{v})_S$ matches the vectors of $V$ one-to-one with the vectors of $\mathbb{R}^n$. Questions about $V$ become questions about $\mathbb{R}^n$.

EXAMPLES 7 AND 8

# Coordinates relative to the standard bases

**$\mathbb{R}^n$, standard basis $S = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$**

$$\mathbf{v} = (v_1, \dots, v_n) = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n, \qquad (\mathbf{v})_S = \mathbf{v}; \qquad \text{e.g. } [(-2,1,3)]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

**$P_n$, standard basis $S = \{1, x, \dots, x^n\}$**

$$\mathbf{p} = c_0 + c_1x + \cdots + c_nx^n, \qquad (\mathbf{p})_S = (c_0, c_1, \dots, c_n); \qquad \text{e.g. } \left(4 - 2x^2 + 3x^3\right)_S = (4,0,-2,3)$$

**$M_{22}$, standard basis $S = \{M_1, M_2, M_3, M_4\}$**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4, \qquad (B)_S = (a, b, c, d); \qquad \text{e.g. } \left(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\right)_S = (5,6,7,8)$$

> Relative to a STANDARD basis the coordinates can be read off. Relative to any other basis they come from solving a linear system: next slide.

EXAMPLE 9

# Coordinates relative to another basis

$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (2,9,0)$, $\mathbf{v}_3 = (3,3,4)$, the basis of Example 3.

**(a) Find $(\mathbf{v})_S$ for $\mathbf{v} = (5,-1,9)$**

Solve $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$: the basis vectors are the columns, $\mathbf{v}$ is the right side.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right] \overset{A_{1,2}^{(-2)},\, A_{1,3}^{(-1)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 5 & -3 & -11 \\ 0 & -2 & 1 & 4 \end{array}\right] \overset{A_{3,2}^{(2)},\, A_{2,3}^{(2)}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

Back-substitution: $c_3 = 2,\ c_2 = -3 + c_3 = -1,\ c_1 = 5 - 2c_2 - 3c_3 = 1$. So $(\mathbf{v})_S = (1,-1,2)$. Check: $(1,2,1) - (2,9,0) + 2(3,3,4) = (5,-1,9)$.

**(b) Find $\mathbf{v}$ if $(\mathbf{v})_S = (-1,3,2)$**

$$\mathbf{v} = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = -(1,2,1) + 3(2,9,0) + 2(3,3,4) = (11,31,7)$$

> Finding coordinates means SOLVING a linear system. Going back from the coordinates to the vector is just a linear combination.

SECTION 4.5

# Dimension

**GOAL** All bases of a space have the same number of vectors; that number is the dimension. Once the dimension is known, deciding whether a set is a basis needs a count and one check.

1. All bases have the same number of vectors
2. Dimension of $\mathbb{R}^n$, $P_n$, $M_{mn}$, of a span and of a solution space
3. The Plus/Minus Theorem
4. $n$ vectors in an $n$-dimensional space
5. Reducing or enlarging a set to a basis
6. Dimensions of subspaces

# Every basis has the same number of vectors

**THEOREM 4.5.2** Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space $V$.

**(a)** A set in $V$ with MORE than $n$ vectors is linearly dependent.

**(b)** A set in $V$ with FEWER than $n$ vectors does not span $V$.

**THEOREM 4.5.1** All bases for a finite-dimensional vector space have the same number of vectors.

## Why Theorem 4.5.1 follows

Let $S'$ be another basis. It is linearly independent, so by (a) it has at most $n$ vectors. It spans $V$, so by (b) it has at least $n$ vectors. Hence it has exactly $n$.

In $\mathbb{R}^n$ this gives Theorem 4.3.3 again: more than $n$ vectors in $\mathbb{R}^n$ are dependent, since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis.

PROOF

# Proof of Theorem 4.5.2(a)

Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in V$ with $m > n$. Each $\mathbf{w}_j$ is a combination of the basis vectors:

$$\mathbf{w}_j = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \cdots + a_{nj}\mathbf{v}_n \qquad (j = 1, \ldots, m)$$

Substitute these into $k_1\mathbf{w}_1 + \cdots + k_m\mathbf{w}_m$ and collect the terms in each $\mathbf{v}_i$:

$$k_1\mathbf{w}_1 + \cdots + k_m\mathbf{w}_m = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n, \qquad c_i = a_{i1}k_1 + a_{i2}k_2 + \cdots + a_{im}k_m$$

The system $c_1 = 0,\ c_2 = 0,\ \ldots,\ c_n = 0$ is homogeneous, with $n$ equations and $m > n$ unknowns $k_1, \ldots, k_m$. So it has a nontrivial solution (Theorem 1.2.2). For that solution,

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0} \quad \text{with } k_1, \ldots, k_m \text{ not all } 0,$$

so $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is linearly DEPENDENT. ■

DEFINITION 1

# Dimension

**DEFINITION 1** The DIMENSION of a finite-dimensional vector space $V$, written $\dim(V)$, is the number of vectors in a basis for $V$. The zero vector space has dimension $0$.

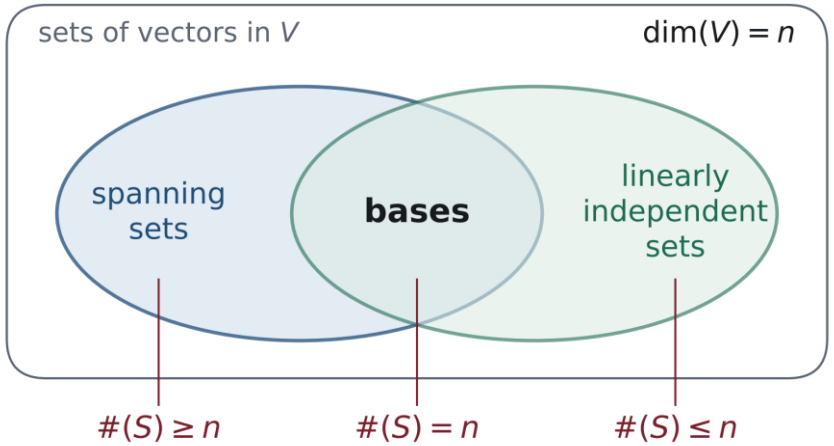
### Example 1: count a standard basis

$\dim(\mathbb{R}^n) = n, \quad \dim(P_n) = n+1, \quad \dim(M_{mn}) = mn.$

### Example 2: the span of an independent set

If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are linearly independent, they are a basis for their span, so $\dim(\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}) = r$.

$P_\infty$ has no finite basis: it is infinite-dimensional.




In a space of dimension $n$: a spanning set has at least $n$ vectors, an independent set at most $n$, a basis exactly $n$.

EXAMPLE 3

# The dimension of a solution space

Find a basis for, and the dimension of, the solution space of

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Section 1.2 (Example 6) found the general solution $x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 0,$ with $r, s, t \in \mathbb{R}$. Pull out the parameters:

$$(x_1, \dots, x_6) = r\,(-3,1,0,0,0,0) + s\,(-4,0,-2,1,0,0) + t\,(-2,0,0,0,1,0)$$

$\mathbf{v}_1 = (-3,1,0,0,0,0)$, $\mathbf{v}_2 = (-4,0,-2,1,0,0)$, $\mathbf{v}_3 = (-2,0,0,0,1,0)$ span the solution space, and they are linearly independent: entries 2, 4 and 5 of $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ are $k_1$, $k_2$ and $k_3$. So they form a basis, and the dimension is 3.

> The parameters of the general solution give a basis: the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is the number of free variables.

EXAMPLE

# The dimension of a subspace: find a basis

**(a)** $W = \{(d,\ c-d,\ c) : c, d \in \mathbb{R}\}$

$$(d,\ c-d,\ c) = c\,(0,1,1) + d\,(1,-1,0), \qquad \text{basis } \{(0,1,1),\ (1,-1,0)\}, \qquad \dim(W) = 2$$

**(b)** $W = \{(2b,\ b,\ 0) : b \in \mathbb{R}\}$

$$(2b,\ b,\ 0) = b\,(2,1,0), \qquad \text{basis } \{(2,1,0)\}, \qquad \dim(W) = 1$$

**(c)** $W = \{A \in M_{22} : A^T = A\}$**, the symmetric** $2\times2$ **matrices**

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \dim(W) = 3$$

In each case the vectors pulled out span $W$, and they are linearly independent: two vectors that are not multiples of each other; one nonzero vector; three matrices whose entries 1 sit in different places.

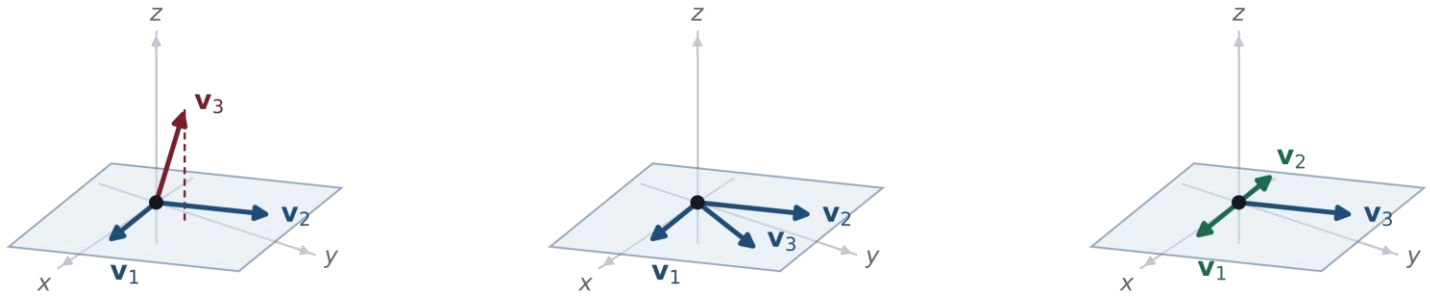
> Recipe: write the general element of $W$, pull out the free parameters, and check that the vectors they multiply are independent. Their number is $\dim(W)$.

# The Plus/Minus Theorem

**THEOREM 4.5.3** (Plus/Minus Theorem) Let $S$ be a nonempty set of vectors in a vector space $V$.

**(a)** If $S$ is linearly independent and $\mathbf{v} \in V$ is outside $\text{span}(S)$, then $S \cup \{\mathbf{v}\}$ is still linearly independent.

**(b)** If $\mathbf{v} \in S$ is a linear combination of the other vectors in $S$, then $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$.




$\mathbf{v}_3$ outside the plane: adjoin it, still independent

three vectors in one plane: remove any one of them

$\mathbf{v}_1, \mathbf{v}_2$ on one line: remove either of them

Example 4. $\mathbf{p}_1 = 1 - x^2$ and $\mathbf{p}_2 = 2 - x^2$ are linearly independent: neither is a multiple of the other. Every linear combination of them has degree at most 2, so $\mathbf{p}_3 = x^3$ is outside their span. By (a), $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent.

THEOREM 4.5.4

# $n$ vectors in an $n$-dimensional space

**THEOREM 4.5.4** Let $\dim(V) = n$, and let $S$ be a set of EXACTLY $n$ vectors in $V$. If $S$ is linearly independent, or if $S$ spans $V$, then $S$ is a basis for $V$.

**TWO OUT OF THREE** For a set $S$ in a vector space of dimension $n$:

**1)** $S$ is linearly independent;

**2)** $S$ spans $V$;

**3)** $S$ has exactly $n$ vectors.

Any two of these give the third, and then $S$ is a basis.

**Example 5: bases by inspection**

- $(-3,7)$ and $(5,5)$: two vectors in $\mathbb{R}^2$, not multiples of each other, so independent, so a basis.

- $(2,0,-1)$, $(4,0,7)$, $(-1,1,4)$: the first two lie in the $xz$-plane and are not multiples of each other; the third is outside the $xz$-plane, so the three are independent (Plus/Minus); three vectors in $\mathbb{R}^3$, so a basis.

THEOREM 4.5.5

# Shrink a spanning set, or enlarge an independent set

**THEOREM 4.5.5** Let $S$ be a finite set of vectors in a finite-dimensional vector space $V$.

**(a)** If $S$ spans $V$ but is not a basis, then $S$ can be REDUCED to a basis for $V$ by removing suitable vectors.

**(b)** If $S$ is linearly independent but not a basis, then $S$ can be ENLARGED to a basis for $V$ by inserting suitable vectors.

## Reduce

$\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (2,0,1)$, $\mathbf{v}_4 = (0,0,-1)$ span a subspace $W$ of $\mathbb{R}^3$. Since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2$, remove them (Plus/Minus (b)). $\{\mathbf{v}_1, \mathbf{v}_2\}$ still spans $W$ and is independent: a basis, and $\dim(W) = 2$. ($W$ is the $xz$-plane.)

## Enlarge

$\mathbf{v}_1 = (1,-2,3)$ and $\mathbf{v}_2 = (0,5,-3)$ are linearly independent. Insert $\mathbf{e}_3 = (0,0,1)$:

$$\det\begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 0 \\ 3 & -3 & 1 \end{bmatrix} = 5 \neq 0$$

so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ is a basis for $\mathbb{R}^3$.

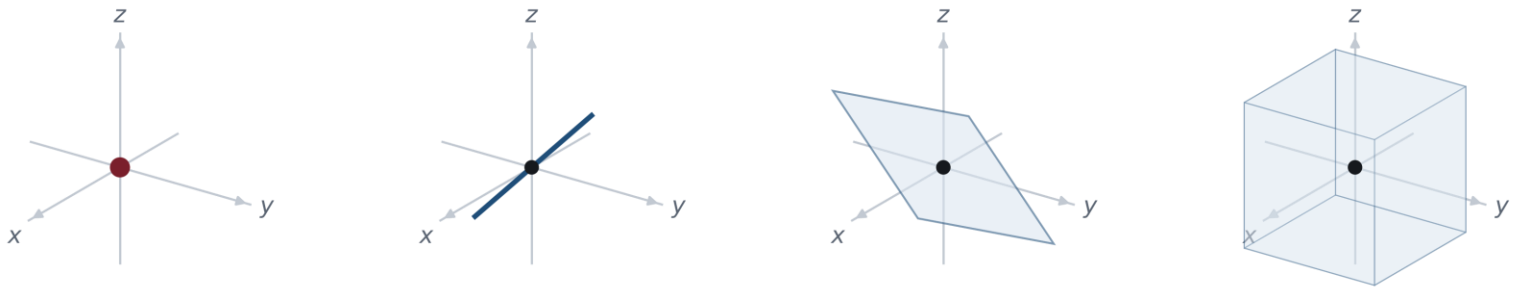
# Subspaces have smaller dimension

**THEOREM 4.5.6** If $W$ is a subspace of a finite-dimensional vector space $V$, then

**(a)** $W$ is finite-dimensional;

**(b)** $\dim(W) \leq \dim(V)$;

**(c)** $W = V$ if and only if $\dim(W) = \dim(V)$.




the origin $\{\mathbf{0}\}$
dimension 0

a line through $\mathbf{0}$
dimension 1

a plane through $\mathbf{0}$
dimension 2

all of $\mathbb{R}^3$
dimension 3

Why (b) and (c): a basis of $W$ is an independent set in $V$. If it does not span $V$, it can be enlarged to a basis of $V$ (Theorem 4.5.5(b)), so $\dim(W) < \dim(V)$; if it spans $V$, then $W = V$.

So the subspaces of $\mathbb{R}^3$ are exactly $\{\mathbf{0}\}$, lines through $\mathbf{0}$, planes through $\mathbf{0}$ and $\mathbb{R}^3$, as listed in Table 1 of Section 4.2.

SECTION 4.6

# Change of Basis

**GOAL** The same vector has different coordinates relative to different bases. One matrix, the transition matrix, converts one coordinate vector into the other.

1. The change-of-basis problem
2. Transition matrices ${}_{B'}P_B$ and ${}_BP_{B'}$
3. Transition matrices are invertible
4. Computing a transition matrix by row reduction
5. Transition to the standard basis of $\mathbb{R}^n$

# Old coordinates from new coordinates

Fix $\mathbf{v} \in V$, and change from an OLD basis $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ to a NEW basis $B' = \{\mathbf{u'}_1, \mathbf{u'}_2\}$. How are $[\mathbf{v}]_B$ and $[\mathbf{v}]_{B'}$ related?

Write the NEW basis vectors in terms of the OLD basis:

$$\mathbf{u'}_1 = a\mathbf{u}_1 + b\mathbf{u}_2, \quad \mathbf{u'}_2 = c\mathbf{u}_1 + d\mathbf{u}_2, \qquad \text{that is,} \qquad [\mathbf{u'}_1]_B = \begin{bmatrix} a \\ b \end{bmatrix}, \quad [\mathbf{u'}_2]_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

If $[\mathbf{v}]_{B'}$ has entries $k_1, k_2$, then

$$\mathbf{v} = k_1\mathbf{u'}_1 + k_2\mathbf{u'}_2 = (ak_1 + ck_2)\,\mathbf{u}_1 + (bk_1 + dk_2)\,\mathbf{u}_2$$

so

$$[\mathbf{v}]_B = \begin{bmatrix} ak_1 + ck_2 \\ bk_1 + dk_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = [\,[\mathbf{u'}_1]_B \;\; [\mathbf{u'}_2]_B\,]\,[\mathbf{v}]_{B'}$$

> The old coordinate vector is a MATRIX times the new one. The columns of that matrix are the new basis vectors, written in coordinates relative to the old basis.

# The transition matrices ${}_{B}P_{B'}$ and ${}_{B'}P_{B}$

For bases $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $B' = \{\mathbf{u'}_1, \ldots, \mathbf{u'}_n\}$ of a vector space $V$:

$$ {}_{B}P_{B'} = [\,[\mathbf{u'}_1]_B \;\; [\mathbf{u'}_2]_B \;\cdots\; [\mathbf{u'}_n]_B\,], \qquad [\mathbf{v}]_B = {}_{B}P_{B'}\,[\mathbf{v}]_{B'} $$

$$ {}_{B'}P_{B} = [\,[\mathbf{u}_1]_{B'} \;\; [\mathbf{u}_2]_{B'} \;\cdots\; [\mathbf{u}_n]_{B'}\,], \qquad [\mathbf{v}]_{B'} = {}_{B'}P_{B}\,[\mathbf{v}]_B $$

**RULE** The columns of the transition matrix FROM one basis TO another are the vectors of the FROM basis, written in coordinates relative to the TO basis.

- Read ${}_{B'}P_{B}$ from right to left, like the formula: it takes the $B$-coordinate vector of $\mathbf{v}$ and returns its $B'$-coordinate vector. In ${}_{B'}P_{B}\,[\mathbf{v}]_B$ the two $B$'s stand side by side.

The textbook writes this matrix as $P_{B \to B'}$.

EXAMPLES 1 AND 2

# Two bases of $\mathbb{R}^2$

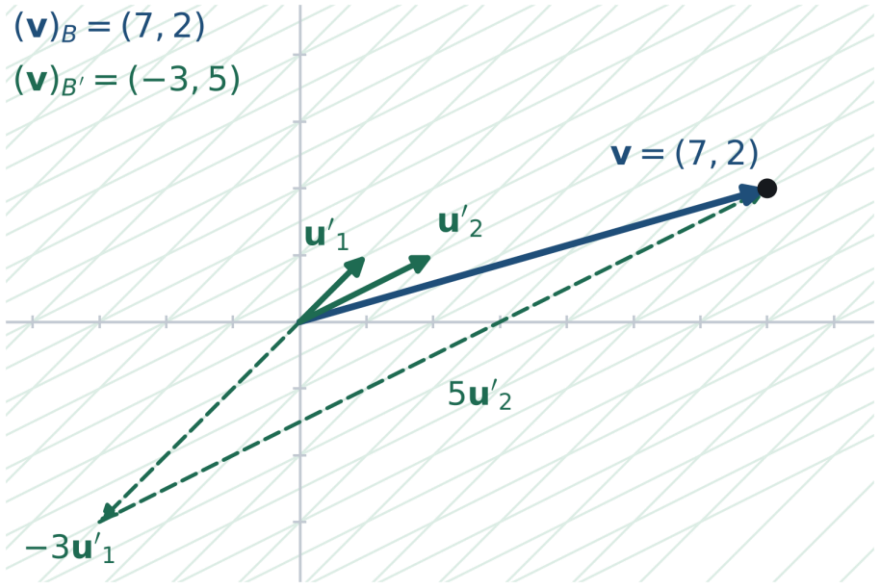
$B = \{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = (1,0)$, $\mathbf{u}_2 = (0,1)$, and $B' = \{\mathbf{u'}_1, \mathbf{u'}_2\}$ with $\mathbf{u'}_1 = (1,1)$, $\mathbf{u'}_2 = (2,1)$.

**(a)** $_BP_{B'}$**: write** $B'$ **in terms of** $B$

$$\mathbf{u'}_1 = \mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{u'}_2 = 2\mathbf{u}_1 + \mathbf{u}_2, \qquad {}_BP_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

**(b)** $_{B'}P_B$**: write** $B$ **in terms of** $B'$

$$\mathbf{u}_1 = -\mathbf{u'}_1 + \mathbf{u'}_2, \quad \mathbf{u}_2 = 2\mathbf{u'}_1 - \mathbf{u'}_2, \qquad {}_{B'}P_B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$




**Example 2:** $(\mathbf{v})_{B'} = (-3,5)$**; find** $[\mathbf{v}]_B$

$$[\mathbf{v}]_B = {}_BP_{B'}\,[\mathbf{v}]_{B'} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

THEOREM 4.6.1

# Transition matrices are invertible

${}_{B'}P_B$ turns $B$-coordinates into $B'$-coordinates, and ${}_{B}P_{B'}$ turns them back. Doing both changes nothing:

$$[\mathbf{v}]_B = {}_{B}P_{B'}\,[\mathbf{v}]_{B'} = {}_{B}P_{B'}\,{}_{B'}P_B\,[\mathbf{v}]_B \quad \text{for every } \mathbf{v} \in V, \qquad \text{so} \qquad {}_{B}P_{B'}\,{}_{B'}P_B = I$$

**THEOREM 4.6.1** If $P$ is the transition matrix from a basis $B'$ to a basis $B$ of a finite-dimensional vector space $V$, then $P$ is invertible, and $P^{-1}$ is the transition matrix from $B$ to $B'$: $\left({}_{B}P_{B'}\right)^{-1} = {}_{B'}P_B$.

Check with Examples 1 and 2:

$${}_{B}P_{B'}\,{}_{B'}P_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So a transition matrix is always square and INVERTIBLE. A matrix with determinant $0$ is never a transition matrix.

PROCEDURE

# Computing a transition matrix in $\mathbb{R}^n$

Finding ${}_{B'}P_B$ means writing each OLD basis vector in terms of the NEW basis: $n$ linear systems, all with the same coefficient matrix. Solve them all at once:

**PROCEDURE** To find ${}_{B'}P_B$ for bases $B$, $B'$ of $\mathbb{R}^n$:

1) Form $[\,B' \mid B\,]$: the NEW basis vectors as columns on the left, the OLD ones on the right.
2) Row reduce until the left side is $I$.
3) The result is $\left[\,I \mid {}_{B'}P_B\,\right]$: read off the transition matrix on the right.

$$[\,\text{new basis} \mid \text{old basis}\,] \overset{\text{row operations}}{\rightarrow} [\,I \mid \text{transition from old to new}\,]$$

**Example 3: Example 1 again, ${}_{B'}P_B$ with $B' = \{(1,1),(2,1)\}$ on the left**

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{A_{1,2}^{(-1)}}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \overset{M_2^{(-1)},\, A_{2,1}^{(-2)}}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

The right side is ${}_{B'}P_B$, as in Example 1(b). The TARGET basis is always on the left, as in the name ${}_{B'}P_B$.

EXAMPLE

# A transition matrix by row reduction

$B = \{(-3,2),\ (4,-2)\}$ and $B' = \{(-1,2),\ (2,-2)\}$ are bases for $\mathbb{R}^2$.

**(a) Find ${}_B P_{B'}$: the target basis $B$ goes on the left**

$$\left[\begin{array}{cc|cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right] \overset{I_{1,2},\ M_1^{(1/2)}}{\rightarrow} \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ -3 & 4 & -1 & 2 \end{array}\right] \overset{A_{1,2}^{(3)},\ A_{2,1}^{(1)}}{\rightarrow} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

**(b) Given $(\mathbf{v})_{B'} = (1,2)$, find $[\mathbf{v}]_B$**

$$[\mathbf{v}]_B = {}_B P_{B'}\,[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

**(c) Find ${}_{B'} P_B$**

$${}_{B'} P_B = \left({}_B P_{B'}\right)^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

Check (b): $1(-1,2) + 2(2,-2) = (3,-2) = -1\,(-3,2) + 0\,(4,-2)$. Check (c): $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = I$.

THEOREM 4.6.2

# Transition to the standard basis

**THEOREM 4.6.2** If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{R}^n$ and $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis, then ${}_SP_B = [\,\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \cdots \;\; \mathbf{u}_n\,]$, the matrix whose columns are the vectors of $B$.

Why: the columns of ${}_SP_B$ are $[\mathbf{u}_1]_S, \ldots, [\mathbf{u}_n]_S$, and $[\mathbf{u}_j]_S = \mathbf{u}_j$ (Section 4.4, Example 7).

## Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = {}_SP_B \qquad \text{for} \qquad B = \{(1,2,1),\ (2,5,0),\ (3,3,8)\}$$

and, by Theorem 4.6.1, ${}_BP_S = A^{-1}$ (computed in Section 1.5, Example 4):

$${}_BP_S = A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}, \qquad [\mathbf{x}]_B = A^{-1}\mathbf{x}$$

Every invertible $n \times n$ matrix is a transition matrix: from the basis formed by its columns to the standard basis.

SUMMARY

# Unit 3B in one slide

### 4.3 Linear independence

- $S$ is linearly independent when $k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$ has ONLY the trivial solution; otherwise some vector is a combination of the others. Sets containing $\mathbf{0}$, and more than $n$ vectors in $\mathbb{R}^n$, are dependent.

### 4.4 Basis and coordinates

- A basis spans $V$ and is linearly independent. Each $\mathbf{v}$ then has exactly one coordinate vector $(\mathbf{v})_S$; finding it means solving a linear system.

### 4.5 Dimension

- All bases have $\dim(V)$ vectors. With exactly $n = \dim(V)$ vectors, independent and spanning go together. Spanning sets shrink to bases, independent sets grow to bases; $\dim(W) \le \dim(V)$ for subspaces.

### 4.6 Change of basis

- $[\mathbf{v}]_{B'} = {}_{B'}P_B[\mathbf{v}]_B$ the columns of ${}_{B'}P_B$ being $[\mathbf{u}_i]_{B'}$; $[\,B' \mid B\,] \to \left[\,I \mid {}_{B'}P_B\,\right]$; and $\left({}_{B'}P_B\right)^{-1} = {}_{B}P_{B'}$.