PHYSICS 507
5th HOMEWORK-Solutions
Prof. V. Lempesis
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1. A ring of radius *a* carries electric charge as follows: the upper half of it is positively uniformly charged with a linear charged density λ. The lower half of it is negatively uniformly charged with a linear charged density $-\lambda$. Find the electric dipole moment of this charge distribution.

Solution:

Consider a part $d\ell$ of the ring at as shown in the figure. The position vector of this part is:

$\mathbf{r} = acos\varphi\mathbf{x} + asin\varphi\mathbf{y}$ (1)
The charge in this part of the ring is $dq = \lambda d\ell = \lambda a d\varphi$. The elementary dipole moment associated with this part is:
$\boldsymbol{d}\mathbf{p} = \mathbf{r}dq$ (2)

Thus the dipole moment of the whole distribution is:

$\mathbf{p} = \int_{ring} \mathbf{r}dq$ (3)

Inserting (1) into (3) we have:

$$\mathbf{p} = a\int_{ring}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})dq$$

$$\mathbf{p} = a\int_0^{\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})dq + a\int_{\pi}^{2\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})dq$$

$$\mathbf{p} = a\int_0^{\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})\lambda a d\varphi - a\int_{\pi}^{2\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})\lambda a d\varphi$$

$$\mathbf{p} = \lambda a^2\int_0^{\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})d\varphi - \lambda a^2\int_{\pi}^{2\pi}(cos\varphi\mathbf{x} + sin\varphi\mathbf{y})\lambda a d\varphi$$

$$\mathbf{p} = \lambda a^2\{\mathbf{x}sin\varphi|_0^{\pi} - \mathbf{y}sin\varphi|_0^{\pi} - \mathbf{x}sin\varphi|_{\pi}^{2\pi} + \mathbf{y}cos\varphi|_{\pi}^{2\pi}\}$$

$$\mathbf{p} = 4\lambda a^2\mathbf{y}$$

2. A dipole **p** is at a distance *r* from a point-like charge *q* and oriented so that **p** makes an angle θ with the vector **r** from *q* to **p**. What is the force on **p**? (Give your answer in full vector form). You are given that the force on a dipole inside an electric field **E** is $\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$.

Solution:

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\mathbf{r} = \frac{1}{4\pi\varepsilon_0}\frac{q(x\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}$$

$$\mathbf{p}\cdot\nabla = p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}$$

Let us calculate the *x*-component of the force:

$$F_x = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\left(\frac{q}{4\pi\varepsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) =$$

$$\frac{q}{4\pi\varepsilon_0}\left[p_x\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) +\right.$$

$$\left.+p_y\frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + p_z\frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\right] =$$

$$\frac{q}{4\pi\varepsilon_0}\left\{p_x\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}x\frac{2x}{(x^2+y^2+z^2)^{5/2}}\right] +\right.$$

$$\left.p_y\left[-\frac{3}{2}x\frac{2y}{(x^2+y^2+z^2)^{5/2}}\right] + p_z\left[-\frac{3}{2}x\frac{2z}{(x^2+y^2+z^2)^{5/2}}\right]\right\} =$$

$$\frac{q}{4\pi\varepsilon_0}\left[\frac{p_x}{r^3} - \frac{3x}{r^5}(p_x x + p_y y + p_z z)\right]$$

Similarly for the other two components so the total force becomes:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}[\mathbf{p} - 3(\mathbf{p}\cdot\mathbf{r})]\mathbf{r}$$

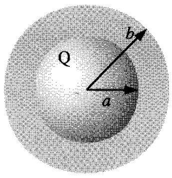
3. A metal sphere of radius *a* carries a charge *Q*. It is surrounded, out to a radius *b*, by a linear dielectric material of permittivity ε and electric susceptibility χ_e.

A) Find the electric field in all regions of space. (Inside the conductor, in the dielectric and in the vacuum) (1 mark).

B) Find the potential at the center of the sphere (relative to infinity) (1 mark).

C) Find the polarization of the dielectric (1 mark).

D) Find the surface bound charge densities in the inner and outer surfaces of the dielectric (1 mark).



Solution:

A) Inside the conductor there is no electric field. Thus **E** = **D** = **P** = **0**.

Also the free charge in the system is Q. Thus applying Gauss' law for the electric displacement **D** or a Gaussian surface of radius r inside the dielectric (i.e. $a < r < b$) we get:

$$\int \mathbf{D} \cdot d\mathbf{S} = Q \Rightarrow \int D\mathbf{r} \cdot (dS\mathbf{r}) = Q \Rightarrow$$

$$D \int dS = Q \Rightarrow D4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}$$

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{r}$$

The electric field is related to the displacement by **E** = **D**/ε. Thus the electric field is given by:

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\varepsilon r^2}\mathbf{r}, & a < r < b \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\mathbf{r}, & r > b \end{cases}$$

B) Thus for the potential we have:

$$V(r) = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{0} (E\mathbf{r}) \cdot (dr'\mathbf{r}) = -\int_{\infty}^{0} Edr$$

but the integral is split into three parts:

$$V(r) = -\int_{\infty}^{0} Edr = -\int_{\infty}^{b} Edr - \int_{b}^{a} Edr - \int_{a}^{0} Edr$$

$$V(r) = -\int_{\infty}^{b} \frac{Q}{4\pi\varepsilon_0 r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2} dr - \int_{a}^{0} 0dr$$

$$V(r) = \frac{Q}{4\pi}\left\{-\frac{1}{\varepsilon_0}\int_{\infty}^{b}\frac{1}{r^2}dr - \frac{1}{\varepsilon}\int_{b}^{a}\frac{1}{r^2}dr\right\}$$

$$V(r) = \frac{Q}{4\pi}\left\{\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right\}$$

C) The polarization of the material is given by:

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \frac{\varepsilon_0\chi_e Q}{4\pi\varepsilon r^2}\mathbf{r}$$

D) The surface bound charge density is given by:

$$\sigma_{\mathrm{b}} = \mathbf{P}\cdot\mathbf{n}|_{\text{surface}} = \begin{cases}\mathbf{P}\cdot\mathbf{n}|_{\text{outer surface}}\\ \mathbf{P}\cdot\mathbf{n}|_{\text{inner surface}}\end{cases}$$

$$\sigma_{\mathrm{b}} = \begin{cases}\mathbf{P}\cdot\mathbf{r}|_{\text{outer surface}}\\ -\mathbf{P}\cdot\mathbf{r}|_{\text{inner surface}}\end{cases}$$

$$\sigma_{\mathrm{b}} = \begin{cases}\dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon b^2}, & \text{(outer surface)}\\ -\dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon a^2} & \text{(inner surface)}\end{cases}$$

Commented [VL1]: You must calculate them on the surfaces so at *r* = *a* and *r*=*b* respectively/