Section 5.6

Substitution in Definite Integrals

The Substitution Formula, and symmetric functions

Math 111 · Integral Calculus

Thomas' Calculus: Early Transcendentals, 15th ed. (Global Edition)

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Where This Lecture Fits

5.4	→	5.5	→	5.6	→	7.1
The Fundamental Theorem		Substitution		Substitution in Definite Integrals		The Logarithm as an Integral
				today		

Today we cover only the first part of §5.6

The Substitution Formula and definite integrals of symmetric functions. The rest of the section — Areas Between Curves — comes later in the course, after Chapter 8, together with volumes and arc length.

Two Ways to Evaluate a Definite Integral by Substitution

METHOD 1

Change the limits

Substitute $u = g(x)$, convert the x-limits into u-limits, and evaluate the new integral in u. You never return to x.

This is Theorem 7.

METHOD 2

Keep the original limits

Find the indefinite integral by substitution, change back to x, then evaluate between the original limits a and b.

This is §5.5 followed by the Evaluation Theorem.

The one mistake to avoid: if you change the variable to u, you must change the limits too. Writing $\int$ from a to b of $f(u)\,du$ with the old x-limits is wrong — the limits belong to the variable you are integrating in.

The Substitution Formula

THEOREM 7

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

1 Same substitution as before

$u = g(x)$ and $du = g'(x)\,dx$ — nothing new in the algebra.

2 New lower limit

the value of u when $x = a$, that is $g(a)$.

3 New upper limit

the value of u when $x = b$, that is $g(b)$.

Example 1 — Method 1: transform the limits

Evaluate $\int 3x^2 \sqrt{(x^3+1)}\,dx$ *from* $x = -1$ *to* $x = 1$

1. **Substitute**

   $$u = x^3 + 1, \qquad du = 3x^2\,dx$$

2. **Convert the limits**

   $$x = -1 \Rightarrow u = (-1)^3 + 1 = 0 \qquad x = 1 \Rightarrow u = 1^3 + 1 = 2$$

3. **Evaluate in u**

   $$\int_0^2 u^{1/2}\,du = (2/3)\,u^{3/2}\,\Big]_0^2$$

4. **Answer**

   $$= (2/3)\cdot 2^{3/2} = 4\sqrt{2}\,/\,3$$

Once the limits are converted, x never appears again — there is nothing to substitute back.

Example 1 — Method 2: keep the original limits

The same integral, worked the other way.

$$\int 3x^2 \sqrt{(x^3 + 1)}\, dx \;=\; \int u^{1/2}\, du \;=\; (2/3)\, u^{3/2} + C$$

$$= \;(2/3)\,(x^3 + 1)^{3/2} + C \qquad \leftarrow \text{change back to } x$$

$$\int_{-1}^{1} 3x^2 \sqrt{(x^3 + 1)}\, dx \;=\; (2/3)\,(x^3 + 1)^{3/2}\,\Big]_{-1}^{1}$$

$$= \;(2/3)\cdot 2^{3/2} - (2/3)\cdot 0 \;=\; 4\sqrt{2}\,/\,3$$

Same answer, as it must be. Method 2 costs one extra step: converting back to x.

Which is better? Here Method 1 is shorter, but that is not always so. Know both, and use whichever suits the integral in front of you.

Try these first

Both are done by transforming the limits. Watch what happens in the second one.

(a) $\int \cot\theta \csc^2\theta \, d\theta$ from $\theta = \pi/4$ to $\theta = \pi/2$ **u = ? new limits = ?**

(b) $\int \tan x \, dx$ from $x = -\pi/4$ to $x = \pi/4$ **u = ? new limits = ?**

For (b), work out the two new limits before you do anything else.

Example 2 (a)

Evaluate $\int \cot\theta \csc^2\theta \, d\theta$ from $\theta = \pi/4$ to $\theta = \pi/2$

$u = \cot\theta, \quad du = -\csc^2\theta \, d\theta \quad \text{so} \quad \csc^2\theta \, d\theta = -du$

$\theta = \pi/4 \Rightarrow u = \cot(\pi/4) = 1 \qquad \theta = \pi/2 \Rightarrow u = \cot(\pi/2) = 0$

$$= -\int_1^0 u \, du$$

the minus sign comes from du

$$= -\left[u^2/2 \right]_1^0$$

integrate in u

$$= -\left[0 - \tfrac{1}{2} \right] = \tfrac{1}{2}$$

and evaluate

Note the limits run from 1 down to 0. Do not reorder them — the sign is already taken care of.

Example 2 (b) — a zero-width interval

Evaluate $\int \tan x\,dx$ from $x = -\pi/4$ to $x = \pi/4$

$\tan x = \sin x / \cos x$. Put $u = \cos x$, $du = -\sin x\,dx$.

$x = -\pi/4 \Rightarrow u = \sqrt{2}/2$ $\qquad$ $x = \pi/4 \Rightarrow u = \sqrt{2}/2$

$$\int_{\sqrt{2}/2}^{\sqrt{2}/2} -du/u = 0$$

Both limits are the same number, so the integral is zero by the zero-width interval rule of §5.3 — **no antiderivative is needed at all.** Converting the limits first saved the whole computation. Geometrically, tan x is odd and the interval is symmetric, which is the subject of the next slide.

Definite Integrals of Symmetric Functions

Even and odd functions on a symmetric interval [−a, a].

THEOREM 8

Let f be continuous on [−a, a].

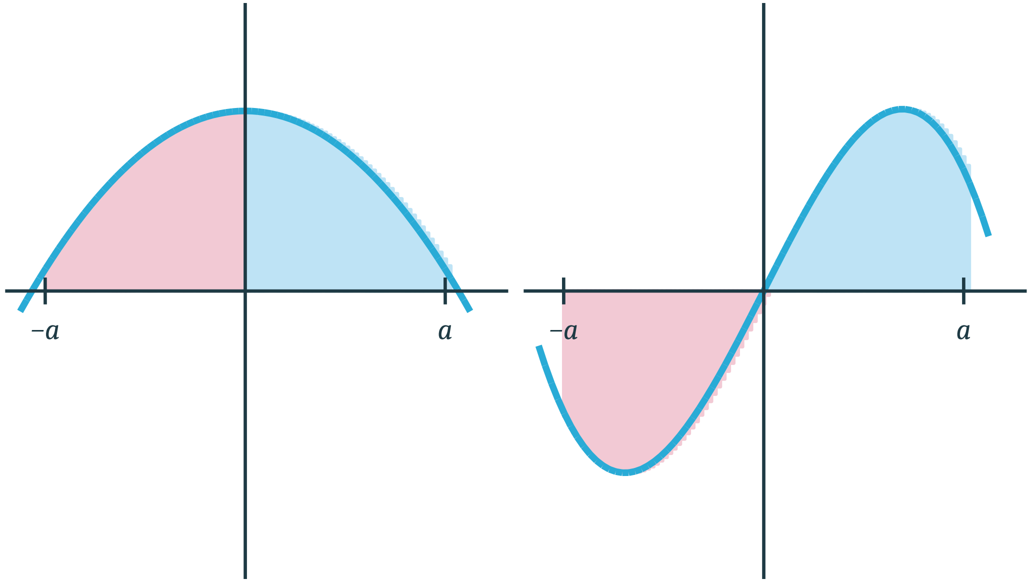
(a) **f even:** $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(b) **f odd:** $\int_{-a}^{a} f(x)\,dx = 0$

Reminders from §1.1

even: $f(-x) = f(x)$ odd: $f(-x) = -f(x)$

Always check the symmetry first: it can halve the work, or finish the problem outright.



(a) f even — the two halves are equal *(b) f odd — the two halves cancel*

Example 3

Evaluate $\int (x^4 - 4x^2 + 6)\, dx$ *from* $x = -2$ *to* $x = 2$

1

Check the symmetry

$$f(-x) = (-x)^4 - 4(-x)^2 + 6 = x^4 - 4x^2 + 6 = f(x)$$

even, and the interval is symmetric

2

Halve the work

$$\int_{-2}^{2} f(x)\, dx = 2 \int_{0}^{2} (x^4 - 4x^2 + 6)\, dx$$

Theorem 8 (a)

3

Integrate

$$= 2 \left[x^5/5 - 4x^3/3 + 6x \right]_0^2$$

only one endpoint to substitute

4

Answer

$$= 2 (32/5 - 32/3 + 12) = 232 / 15$$

the lower limit contributes nothing

Without the symmetry you would evaluate the bracket at both −2 and 2 and then subtract — more arithmetic, more chances to slip.

Summary

1. **Theorem 7** — ∫ from a to b of f (g (x)) g ′(x) dx = ∫ from g (a) to g (b) of f (u) du.

2. **Change the variable, change the limits** — the new limits are the values of u at x = a and x = b.

3. **Or keep the original limits** — find the antiderivative, convert back to x, then evaluate. Both are correct.

4. **Equal limits give zero** — if g (a) = g (b), the integral is 0 — no antiderivative required.

5. **Theorem 8** — on [−a, a]: an even function gives twice the integral from 0 to a; an odd function gives 0.

Practice

Exercises 5.6

1–48 evaluate using the Substitution Formula

Areas between curves come later in the course, with Chapter 6.

Next lecture: §7.1 — The Logarithm Defined as an Integral