Section 5.5

Indefinite Integrals and the Substitution Method

Running the Chain Rule backwards

Math 111 · Integral Calculus

Thomas' Calculus: Early Transcendentals, 15th ed. (Global Edition)

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Where This Lecture Fits

4.8	→	5.4	→	5.5	→	5.6
Antiderivatives		The Fundamental Theorem		Substitution		Substitution in Definite Integrals
				today		

The problem we now face

The Fundamental Theorem turns a definite integral into an antiderivative — but only if we can find one. So far we could only integrate functions we already recognised as derivatives.

What substitution gives us

A way to turn an integral we cannot do into one we can, by reversing the Chain Rule. It is the first real technique of integration in this course.

Definite and Indefinite Integrals — Keep Them Apart

DEFINITE

$$\int_a^b f(x)\,dx$$

is a number.

It has limits of integration, and the Fundamental Theorem evaluates it as $F(b) - F(a)$.

INDEFINITE

$$\int f(x)\,dx = F(x) + C$$

is a function, plus an arbitrary constant.

It has no limits. It denotes the whole set of antiderivatives of f, as defined in §4.8.

Why the same symbol is used for both

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(b) + C] - [F(a) + C] = \left[\,F(x) + C\,\right]_a^b$$

The constant cancels, so evaluating the indefinite integral between a and b gives the definite integral. That is why both carry the sign $\int$.

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and $n \neq -1$, the Chain Rule says:

$$\frac{d}{dx}\left[\frac{u^{n+1}}{n+1} \right] = u^n \cdot \frac{du}{dx}$$

Read backwards, that says $u^{n+1}/(n+1)$ is an antiderivative of $u^n \,(du/dx)$, so

$$\int u^n \left(\frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C \quad \text{— which is exactly } \int u^n \, du$$

The key move

$$du = \left(\frac{du}{dx}\right) dx$$

The differential from §3.11 — it lets us swap $(du/dx)\,dx$ for du.

Leibniz saw it first

His notation was built so that this substitution would work as ordinary algebra — the dx is not decoration, it is what makes the method possible.

Example 1

Find $\int (x^3 + x)^5 (3x^2 + 1)\,dx$

1. **Spot the inside function**

   $u = x^3 + x$

   the factor $3x^2 + 1$ *that multiplies it is exactly its derivative*

2. **Take the differential**

   $du = (3x^2 + 1)\,dx$

   so the whole tail of the integrand becomes du

3. **Rewrite and integrate**

   $\int u^5\,du = u^6/6 + C$

   an integral we can do at sight

4. **Put x back**

   $= (x^3 + x)^6/6 + C$

   never leave the answer in terms of u

Check: differentiate $(x^3 + x)^6/6$ *by the Chain Rule and you get the integrand back.*

Example 2 — when a constant is missing

Find $\int \sqrt{(2x + 1)}\, dx$

The obstacle

$$u = 2x + 1 \;\Rightarrow\; du = 2\, dx$$

But the integral has only dx, not 2 dx. The constant factor 2 is missing.

The fix

$$\int \sqrt{(2x+1)}\, dx \;=\; \tfrac{1}{2} \int \sqrt{(2x+1)} \cdot 2\, dx$$

Insert the 2 you need and put ½ in front to compensate. This works only for constants — never for a factor involving x.

$$= \tfrac{1}{2} \int u^{1/2}\, du$$

Take out the 2

$$= \tfrac{1}{2} \cdot (2/3)\, u^{3/2} + C$$

$$= \tfrac{1}{3}\, (2x + 1)^{3/2} + C$$

The Substitution Rule

THEOREM 6

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

The method, in three steps:

1 Substitute

$u = g(x)$ and $du = g'(x)\, dx$

to turn the integral into $\int f(u)\, du$

2 Integrate

with respect to u

this is the step that has to be easy — if it is not, try another u

3 Replace

u by $g(x)$

the answer must be a function of x

Try these first

Choose u, write down du, and see what the integral becomes.

(a) $\int \sec^2(5x + 1) \cdot 5\,dx$ **$u = ?$**

(b) $\int \cos(7\theta + 3)\,d\theta$ **$u = ?$**

(c) $\int x\sqrt{(2x + 1)}\,dx$ **$u = ?$**

One of these has the derivative sitting right there. One needs a constant put in by hand. One has an extra x that will not go away by itself.

Example 3

Find $\int \sec^2(5x + 1) \cdot 5\,dx$

$u = 5x + 1, \quad du = 5\,dx$ — the 5 is already in the integrand

$$\int \sec^2 u\,du$$ *the integral in terms of u*

$$= \tan u + C$$ *since $d/du\,(\tan u) = \sec^2 u$*

$$= \tan(5x + 1) + C$$ *put x back*

This is the easiest case: the derivative of the inside function is already a factor of the integrand.

Example 4 — two ways to fix the constant

Find $\int \cos(7\theta + 3)\,d\theta$, *with* $u = 7\theta + 3$, $du = 7\,d\theta$

MULTIPLY AND DIVIDE

$$\int \cos(7\theta + 3)\,d\theta$$

$$= \tfrac{1}{7} \int \cos(7\theta + 3) \cdot 7\,d\theta$$

$$= \tfrac{1}{7} \int \cos u\,du$$

$$= \tfrac{1}{7} \sin u + C$$

SOLVE FOR dθ

$$du = 7\,d\theta \;\Rightarrow\; d\theta = \tfrac{1}{7}\,du$$

$$\int \cos(7\theta + 3)\,d\theta$$

$$= \int \cos u \cdot \tfrac{1}{7}\,du$$

$$= \tfrac{1}{7} \sin u + C$$

Either way: $\int \cos(7\theta + 3)\,d\theta = \tfrac{1}{7} \sin(7\theta + 3) + C$

Example 6 — an extra factor of x

Evaluate $\int x \sqrt{(2x + 1)}\, dx$

The same substitution as Example 2 leaves an x behind that will not cancel. The way out: solve the substitution equation for x.

$u = 2x + 1, \quad du = 2\, dx$	*as before*
$x = (u - 1)/2$	*solve the substitution equation for x*
$\int x \sqrt{(2x+1)}\, dx = \tfrac{1}{4} \int (u - 1)\, u^{1/2}\, du$	*everything is now in u*
$= \tfrac{1}{4} \int (u^{3/2} - u^{1/2})\, du$	*multiply through before integrating*
$\mathbf{= \tfrac{1}{10} (2x + 1)^{5/2} - \tfrac{1}{6} (2x + 1)^{3/2} + C}$	*integrate, then put x back*

Example 7 — change the integrand first

Trigonometric identities can turn an integral we cannot do into one we can.

Half-angle identities: $\sin^2 x = (1 - \cos 2x)/2$ $\quad$ $\cos^2 x = (1 + \cos 2x)/2$

(a) $\int \sin^2 x\, dx = \frac{1}{2} \int (1 - \cos 2x)\, dx = x/2 - (\sin 2x)/4 + C$

(b) $\int \cos^2 x\, dx = \frac{1}{2} \int (1 + \cos 2x)\, dx = x/2 + (\sin 2x)/4 + C$

Neither integrand yields to a substitution as it stands — the identity is what makes cos 2x integrable directly.

Trying Different Substitutions — Example 9

Evaluate $\int 2z\,dz / \sqrt[3]{z^2+1}$. More than one choice of u can work.

METHOD 1: $u = z^2 + 1$

$du = 2z\,dz$

$\int du / u^{1/3} = \int u^{-1/3}\,du$

$= (3/2)\,u^{2/3} + C$

$= \mathbf{(3/2)\,(z^2 + 1)^{2/3} + C}$

METHOD 2: $u = \sqrt[3]{z^2 + 1}$

$u^3 = z^2 + 1, \quad 3u^2\,du = 2z\,dz$

$\int 3u^2\,du / u = 3\int u\,du$

$= (3/2)\,u^2 + C$

$= \mathbf{(3/2)\,(z^2 + 1)^{2/3} + C}$

Same answer. Substitution is an exploratory tool: try the most troublesome part of the integrand first, and if that leads nowhere, try something else. The skill comes from practice, not from a rule.

Summary

1. **Two different objects** — $\int f\,dx$ from a to b is a number; $\int f\,dx$ is a function plus C.
2. **Theorem 6** — $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$ — the Chain Rule read backwards.
3. **Three steps** — substitute u and du, integrate in u, then replace u by $g(x)$.
4. **Missing constants** — insert the constant you need and compensate in front — constants only, never a factor with x in it.
5. **Extra factors of x** — solve $u = g(x)$ for x and substitute that too.
6. **No single rule** — if a substitution leads nowhere, try another, or change the integrand with an identity first.

Practice

Exercises 5.5

1–16 substitution given
17–66 evaluate the integral
67–68 two-step substitutions
73–78 initial value problems

Differentiate every answer. It is the fastest way to catch a wrong u.

Next lecture: §5.6 — Substitution in Definite Integrals