Section 5.4

The Fundamental Theorem of Calculus

The central theorem of integral calculus

Math 111 · Integral Calculus

Thomas' Calculus: Early Transcendentals, 15th ed. (Global Edition)

Department of Mathematics, King Saud University

Where This Lecture Fits

5.3	→	4.8	→	5.4	→	5.5
The Definite Integral		Antiderivatives		The Fundamental Theorem		Substitution Method
				today		

Two strands of calculus meet here

One strand is the limit of finite sums that gave us the definite integral (5.2, 5.3). The other is derivatives and antiderivatives (Ch. 3, 4.8). The Fundamental Theorem ties them together.

What it buys us

A definite integral can be evaluated from an antiderivative at just two points — no partitions, no limits of Riemann sums. Newton and Leibniz built two centuries of science on this.

Mean Value Theorem for Definite Integrals

The quantity below is the average (mean) value of f on [a, b]. The theorem says a continuous function actually attains its average value somewhere on the interval.

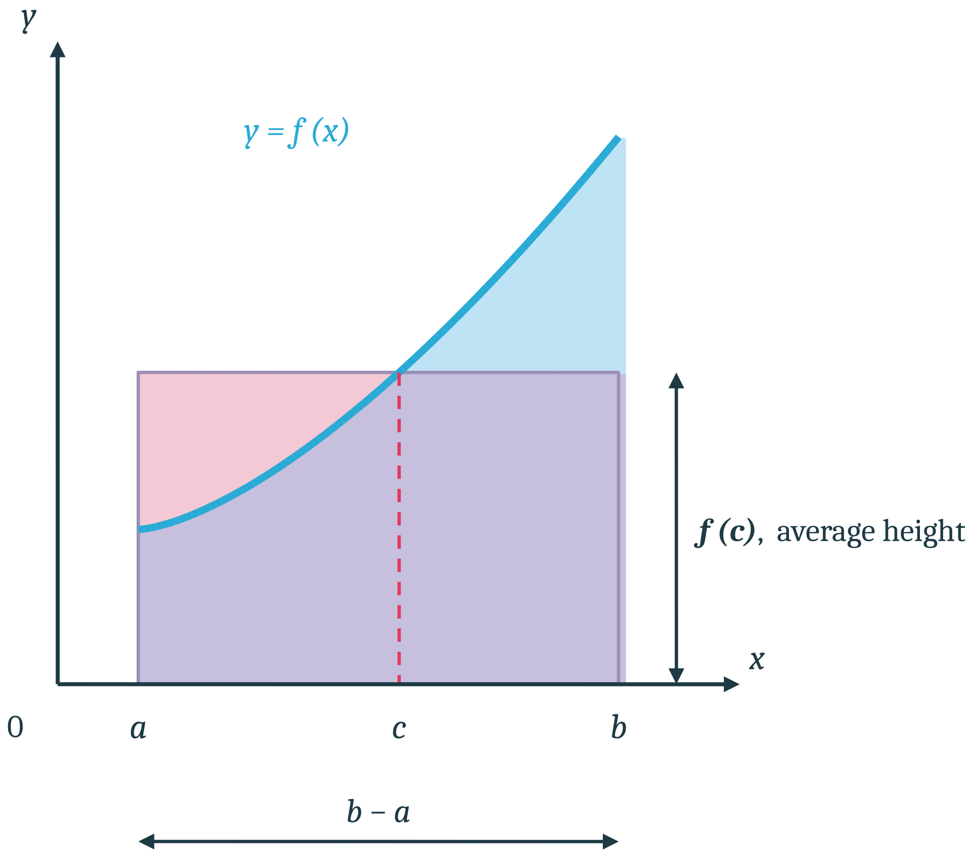
THEOREM 3

If f is continuous on [a, b], then at some point c in [a, b],

$$f(c) = 1/(b-a) \cdot \int_a^b f(x)\,dx$$

— the average value of f on [a, b].

Geometrically: the rectangle of height f (c) and base b − a has exactly the same area as the region under the graph of f from a to b.



The pink area the rectangle adds on the left equals the blue area the graph adds on the right.

Why Continuity Is Essential

Theorem 3 relies on the Intermediate Value Theorem, which needs continuity. Drop it and the conclusion can fail.

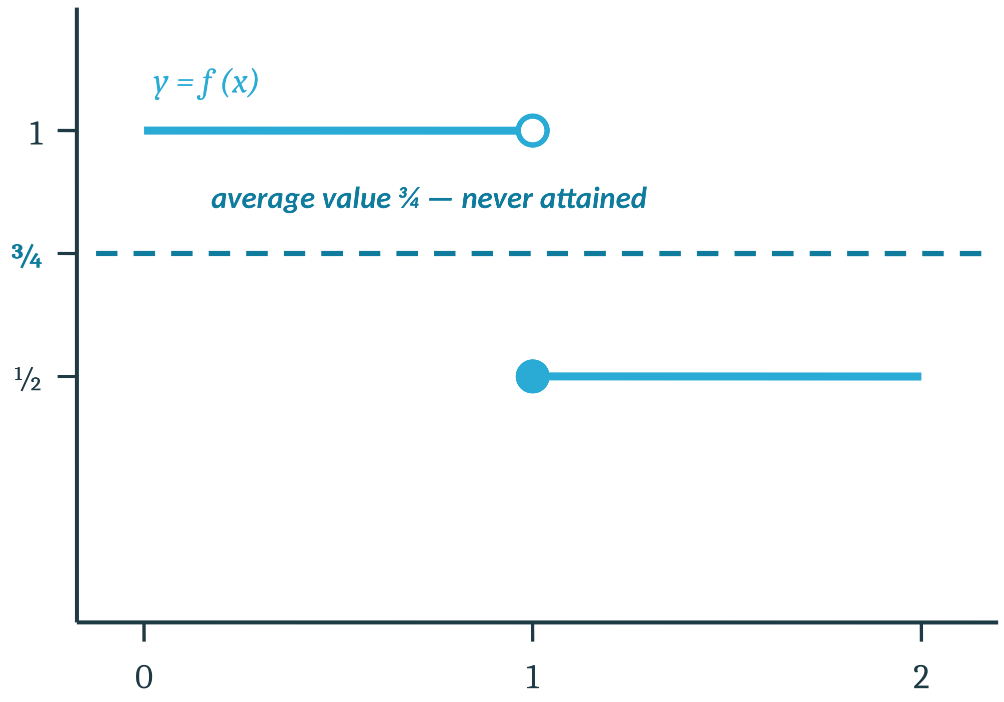
A discontinuous counterexample

$$f(x) = 1 \text{ on } [0,1) \quad \text{and} \quad f(x) = \tfrac{1}{2} \text{ on } [1,2]$$

$$\textbf{average value} = \tfrac{1}{2} \cdot \int_0^2 f(x)\,dx = \tfrac{3}{4}$$

But f only ever takes the values 1 and ½ — it never equals ¾. No such c exists.

Where continuity is used: the Intermediate Value Theorem needs it here, and Part 1 of the Fundamental Theorem needs it again. Every statement in this lecture assumes f is continuous on the interval.



The graph jumps straight over the value ¾ — there is no c with f (c) = ¾.

Average Value — Example 6 (§5.3)

The quantity Theorem 3 is about, and a worked case.

DEFINITION

If f is integrable on [a, b], its average value (or mean) is

$$f_{av} = 1/(b-a)\cdot\int_a^b f(x)\,dx$$

Example 6 — Find the average value of f (x) = x on [1, 3].

1. **Evaluate the integral**

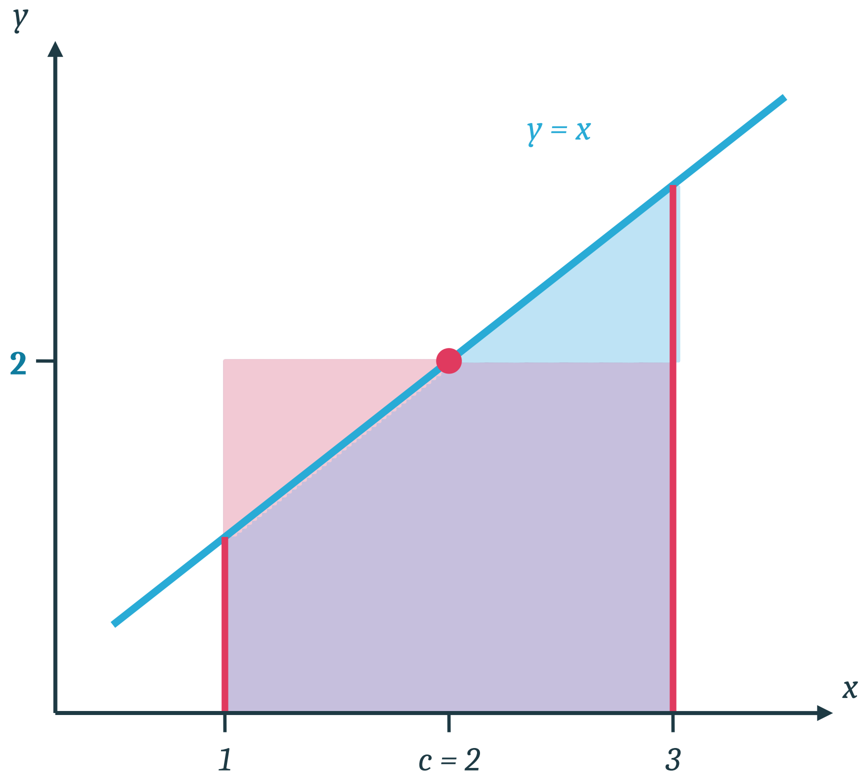
   $$\int_1^3 x\,dx = 9/2 - 1/2 = 4$$

2. **Divide by the length of the interval**

   $$f_{av} = (1/2)\cdot 4 = 2$$

3. **Read it through Theorem 3**

   f (c) = 2 at c = 2, so the average value really is attained.



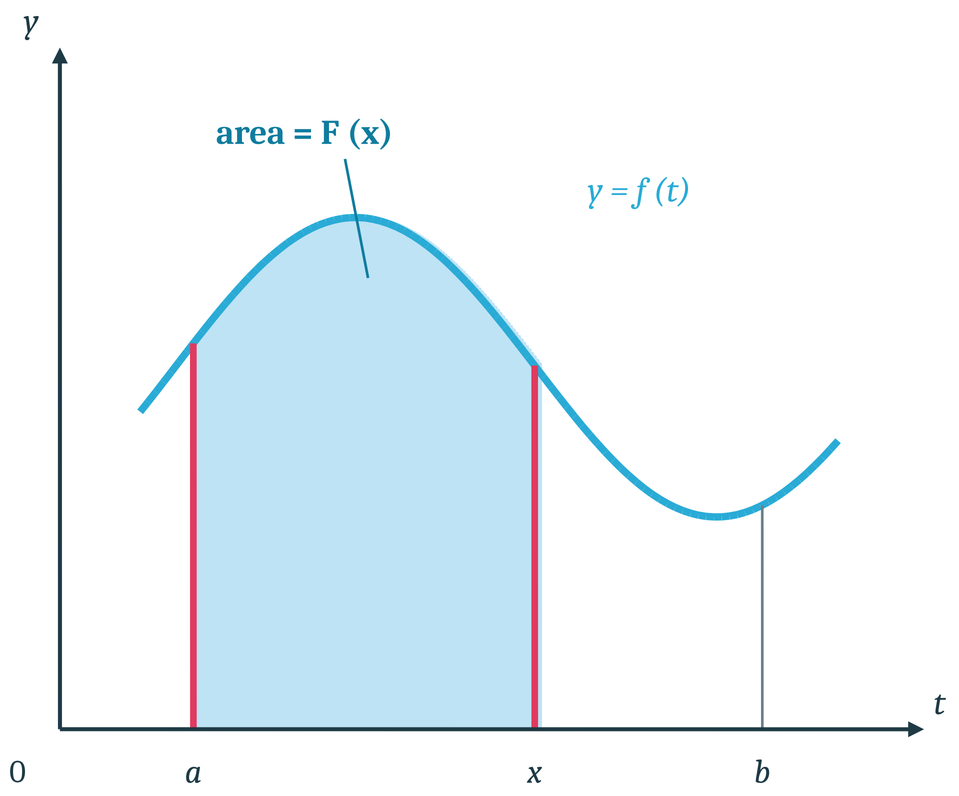
The rectangle of height 2 over [1, 3] has area 4 — the same as the region under y = x.

A New Function Built from an Integral

Fix a in an interval I and let the upper limit vary. For each x in I this produces one number, so it defines a function:

$$F(x) = \int_a^x f(t)\,dt$$

1. **x is the upper limit —** It is the input of F. Every input gives one output — F is an ordinary function.
2. **t is a dummy variable —** It only lives inside the integral. Writing $\int f(u)\,du$ would define the same F.
3. **Geometric reading —** If $f \geq 0$ and $x > a$, then $F(x)$ is the area under the graph of f from a to x.



For $f \geq 0$ and $x > a$, $F(x)$ is the shaded area under the graph from a to x.

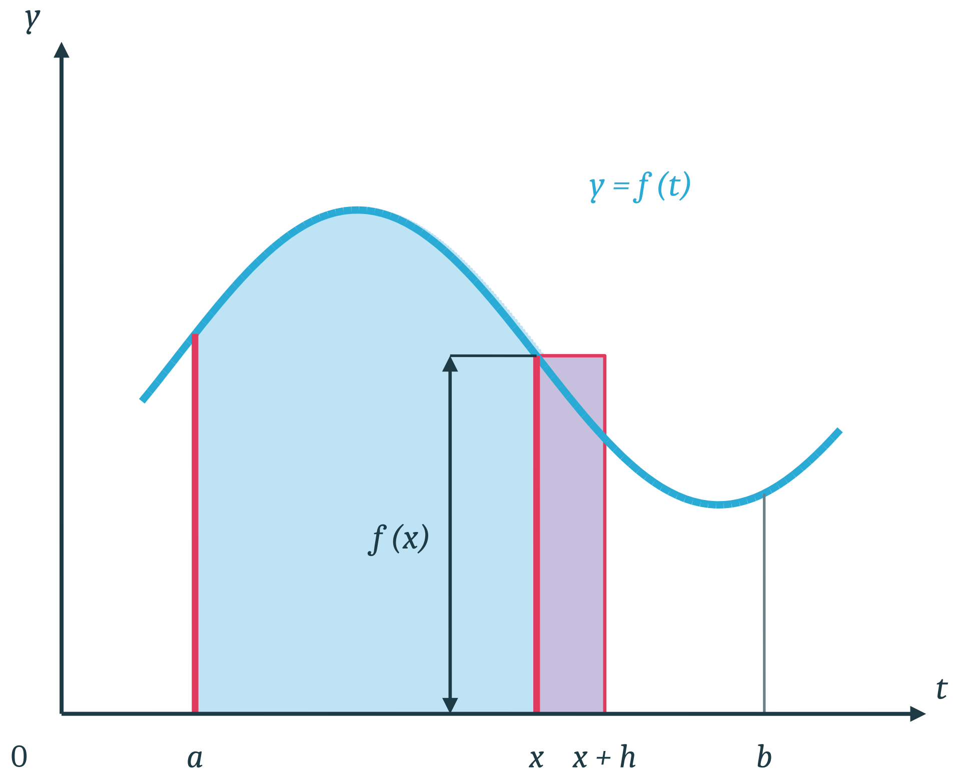
Why the Derivative of F Is f Itself

The geometric idea behind the theorem, for $f \geq 0$.

$F(x+h) - F(x)$	is the area under the graph between x and $x + h$.
$\approx h \cdot f(x)$	for small h that sliver is nearly a rectangle of base h and height $f(x)$.
$[F(x+h) - F(x)] / h \approx f(x)$	divide by h — the difference quotient is close to $f(x)$.

Letting $h \to 0$ turns the approximation into an equality:

$$F'(x) = \lim_{h\to 0} [F(x+h) - F(x)] / h = f(x)$$



The narrow band between x and $x + h$ has area $\approx h \cdot f(x)$.

The Fundamental Theorem, Part 1

THEOREM 4 — PART 1

If f is continuous on [a, b], then F (x) = ∫ f (t) dt from a to x is continuous on [a, b], differentiable on (a, b), and its derivative is f itself:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

1 Differentiation undoes integration

Integrate f, then differentiate, and f comes back unchanged.

2 Every continuous f has an antiderivative

Namely F itself. This is what makes Part 2 possible.

3 Watch the upper limit

If it is anything other than x — say x^2 — the Chain Rule is needed.

Example 2 — Try these first

Part 1 of the Fundamental Theorem is all you need. Watch where x sits in each integral.

Find dy/dx in each case.

(a) $y = \int_a^x (t^3 + 1)\, dt$ **dy/dx = ?**

(b) $y = \int_x^5 3t \sin t\, dt$ **dy/dx = ?**

(c) $y = \int_1^{x^2} \cos t\, dt$ **dy/dx = ?**

Two of these need more than a direct reading of Equation (2) — decide which, and why.

Example 2 — Find dy/dx

In each part x sits in a limit of integration; t is only a dummy variable.

(a) $y = \int_a^x (t^3 + 1)\,dt \longrightarrow$ $\mathbf{dy/dx = x^3 + 1}$

Direct use of Equation (2)

(b) $y = \int_x^5 3t \sin t\,dt \longrightarrow$ $\mathbf{dy/dx = -3x \sin x}$

Flip the limits first (Rule 1), which changes the sign

(c) $y = \int_1^{x^2} \cos t\,dt \longrightarrow$ $\mathbf{dy/dx = 2x \cos x^2}$

Upper limit is x^2, so put $u = x^2$ and use the Chain Rule

∫ Where the Sign ∫ Comes From

A short break — the notation you have been writing all lecture is 350 years old.

1. **29 October 1675**

   Working alone in his Paris lodgings, Gottfried Wilhelm Leibniz wrote ∫ for the first time, in an unpublished manuscript.

2. **It is the letter s**

   ∫ is a long s — the old handwritten form of the letter — standing for summa, Latin for sum. Leibniz pictured the integral as a sum of infinitely many small pieces.

3. **Before that, he wrote "omn."**

   Short for omnia, "all of". Two weeks later, on 11 November 1675, he added the dx we still write after the integrand.

4. **Newton had it too**

   He reached the same theorem independently but wrote a small bar above the variable. Leibniz's notation survived because it was easier to compute with.

Analyseos tetragonisticae pars secunda

Paris, 29 October 1675

ſ → ∫

long s *summa*

$$\int l \;=\; \text{omn.}\, l$$

"the sum of all the l's"

First in print: Acta Eruditorum, June 1686

So every ∫ you write is a seventeenth-century letter s, still doing the job Leibniz gave it: adding up infinitely many pieces.

The Fundamental Theorem, Part 2 — The Evaluation Theorem

THEOREM 4 — PART 2

If f is continuous over [a, b] and F is any antiderivative of f on [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

To evaluate a definite integral, do only two things:

1. Find any antiderivative F of f.
2. Compute the number F (b) − F (a).

No partitions, no Riemann sums — only two values of F.

Notation for F (b) − F (a)

$$F(x)\,\Big]_a^b \quad \text{or} \quad \Big[F(x)\Big]_a^b$$

Use the square brackets when F has more than one term. Any antiderivative works — the constant C cancels in the subtraction.

Example 3 — Evaluating definite integrals

Compare these with the Riemann-sum computations of §5.3.

(a) $\int_0^{\pi} \cos x \, dx$ $= \sin x \,\Big]_0^{\pi} = \sin \pi - \sin 0 = 0$ *since $d/dx \sin x = \cos x$*

(b) $\int_0^{\pi/4} \sec x \tan x \, dx$ $= \sec x \,\Big]_0^{\pi/4} = \sqrt{2} - 1$ *since $d/dx \sec x = \sec x \tan x$*

(c) $\int_1^4 \left((3/2)\sqrt{x} - 4/x^2 \right) dx$ $= \left[x^{3/2} + 4/x \right]_1^4 = 9 - 5 = 4$ *check by differentiating the bracket*

Every one of these took a single line. This is the whole point of the theorem.

Integration and Differentiation Are Inverse Processes

INTEGRATE, THEN DIFFERENTIATE

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

You get f back, exactly.

DIFFERENTIATE, THEN INTEGRATE

$$\int_a^x F'(t)\,dt = F(x) - F(a)$$

You get F back, up to a constant.

Three consequences worth stating explicitly:

1. Every continuous function has an antiderivative.
2. Finding antiderivatives is the key skill for evaluating definite integrals — which is why §4.8 came first.
3. The differential equation $dy/dx = f(x)$ always has a solution when f is continuous, namely $y = F(x) + C$.

∫ Total Area — Example 6

Area is never negative, but a definite integral can be.

Take $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ on $[-2, 2]$.

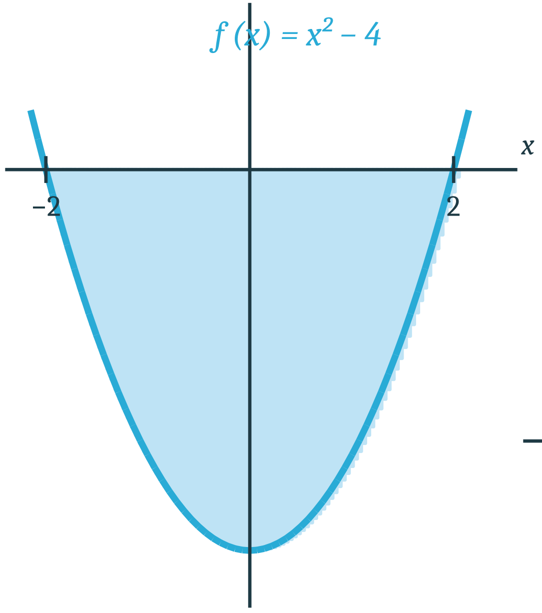
$$\int_{-2}^{2} f(x)\,dx = \left[x^3/3 - 4x \right]_{-2}^{2} = -32/3$$

$$\int_{-2}^{2} g(x)\,dx = \left[4x - x^3/3 \right]_{-2}^{2} = +32/3$$

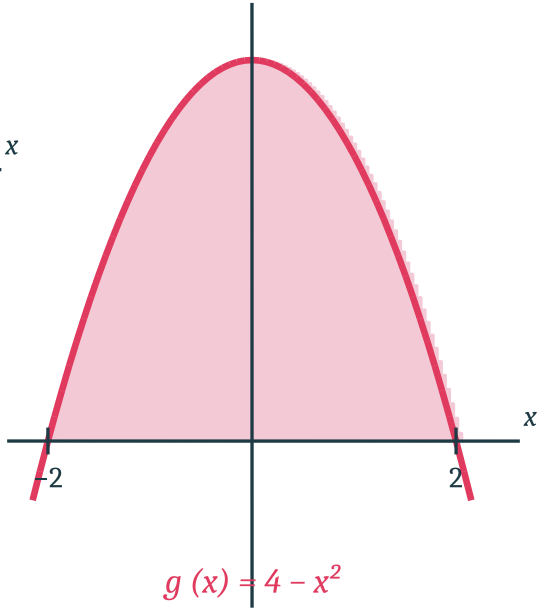
In both cases the area between the graph and the x-axis is 32/3 square units.

To find the area between $y = f(x)$ and the x-axis on $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.



$\int f(x)\,dx = -32/3$



$\int g(x)\,dx = +32/3$

Same shaded area on each side, but the integrals differ in sign.

Summary

1. **Theorem 3** — a continuous function attains its average value: $f(c) = 1/(b - a) \int f$, for some c in $[a, b]$.
2. **Part 1** — $d/dx \int$ from a to x of $f(t)\,dt = f(x)$. Differentiation undoes integration, and every continuous f has an antiderivative.
3. **Chain Rule** — when the upper limit is not x itself, substitute u and differentiate through.
4. **Part 2** — $\int$ from a to b of $f = F(b) - F(a)$ for any antiderivative F. Two evaluations replace a limit of Riemann sums.
5. **Total area** — split at the zeros of f, integrate on each piece, then add absolute values.

Practice

Exercises 5.4

1–34 evaluating integrals

39–44 derivatives of integrals

45–56 finding dy/dx

57–64 total area

Check each answer by differentiating the antiderivative you used.

Next lecture: §5.5 — Indefinite Integrals and the Substitution Method