SOLUTION MID TERM II, Nov 5, 2024
Dept. Math., College of Science, KSU
Math: 107 Full Mark: 25 Time: 90 Minutes

Q1. [3+3=6] (a) $\overrightarrow{AB} = \langle -3, -2, 2\rangle$, $\overrightarrow{AC} = \langle -2, 2, 3\rangle$. It follows that $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -10, 5, -10\rangle$. 1.5
Then the area of the triangle is $A = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{15}{2}$. 1.5
(b) The distance $d = \frac{\|\overrightarrow{AB}\times\overrightarrow{AC}\|}{\|\overrightarrow{AB}\|} = \frac{15}{4.12} \approx 3.64$. 2+1

Q2. [3+4=7] (a) The intersection of the planes P_1 and P_2 is a straight line l (say). Since the vectors $\mathbf{n}_1 = \langle 1, -2, 3\rangle$ and $\mathbf{n}_2 = \langle 1, 1, 1\rangle$ are normal to the planes P_1 and P_2, respectively, their cross-product $\mathbf{n}_1 \times \mathbf{n}_2 = \langle -5, 2, 3\rangle$ must be parallel to the line l. Clearly, $(1, 0, 0)$ being a common solution of the given equations of both the planes P_1 and P_2 lies on the intersection line l. Hence, the parametric equations of the line l are as follows: 1

$$x = 1 - 5t,\ y = 2t,\ z = 3t, \text{ where } t \in \Re.$$ 2

(b) We consider the following system of three equations in two variables

$$\left.\begin{aligned} 1 + 2t &= 4 - s \\ 1 - 4t &= -1 + 6s \\ 5 - t &= 4 + s. \end{aligned}\right\}$$

$\Longrightarrow$

$$\left.\begin{aligned} 2t + s &= 3 \\ 4t + 6s &= 2 \\ t + s &= 1. \end{aligned}\right\}$$ 2

Solving first two equations, we obtain $t = 2$ and $s = -1$. The point of intersection can be found by letting t for l_1 or by letting s for l_2. In either case, we obtain $P(5, -7, 3)$. 2

Q3. [3] The given equation of the surface can be written as $\frac{x^2}{1^2} + \frac{y^2}{4^2} - \frac{z^2}{8^2} = 1$, which is a hyperboloid of one sheet. 1+1+1
xy-trace is $\frac{x^2}{1^2} + \frac{y^2}{4^2} = 1$, which is an ellipse; yz-trace is $\frac{y^2}{4^2} - \frac{z^2}{8^2} = 1$, which is a hyperbola; xz-trace is $\frac{x^2}{1^2} - \frac{z^2}{8^2} = 1$, which is a hyperbola. Sketch is given at the last page.

Q4. [5+4=9] (a) Given $\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t^2\mathbf{k}$. Then
velocity $\mathbf{v}(t) = \mathbf{r}'(t) = [\cos t - t\sin t]\mathbf{i} + [\sin t + t\cos t]\mathbf{j} + 2t\mathbf{k}$.
Acceleration $\mathbf{a}(t) = \mathbf{r}''(t) = [-2\sin t - t\cos t]\mathbf{i} + [2\cos t - t\sin t]\mathbf{j} + 2\mathbf{k}$. 2
$\mathbf{v}(\frac{\pi}{2}) = -\frac{\pi}{2}\mathbf{i}+\mathbf{j}+\pi\mathbf{k}$; speed $\|\mathbf{v}(\frac{\pi}{2})\| = \sqrt{\frac{5\pi^2+4}{4}} = \frac{1}{2}\sqrt{5\pi^2+4}$, and acceleration $\mathbf{a}(\frac{\pi}{2}) = -2\mathbf{i}-\frac{\pi}{2}\mathbf{j}+2\mathbf{k}$. 1+1+1
(b) $\mathbf{r}''(t) = 6t\mathbf{i} + 3\mathbf{j}$ implies $\mathbf{r}'(t) = 3t^2\mathbf{i} + 3t\mathbf{j} + \mathbf{c}_1$.
$\mathbf{r}(0) = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{r}'(0) = \mathbf{c}_1$ implies $\mathbf{c}_1 = \mathbf{r}'(0)$. Hence $\mathbf{r}'(t) = (3t^2 + 4)\mathbf{i} + (3t - 1)\mathbf{j} + \mathbf{k}$. and $\mathbf{r}(t) = (t^3 + 4t)\mathbf{i} + (\frac{3}{2}t^2 - t)\mathbf{j} + t\mathbf{k} + \mathbf{c}_2$. 2
$\mathbf{r}(0) = 7\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{c}_2$ implies $\mathbf{c}_2 = \mathbf{r}(0)$. Thus, $\mathbf{r}(t) = (t^3 + 4t)\mathbf{i} + (\frac{3}{2}t^2 - t + 7)\mathbf{j} + t\mathbf{k}$. 2

Sketch of the given surface