1st Mid Exam

Answer the following questions:

1	Given a vector $\mathbf{t} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$, show, with the help of Stokes' theorem, that the integral around a continuous closed curve in the xy-plane $\frac{1}{2}\oint \mathbf{t} \cdot d\boldsymbol{\lambda} = \frac{1}{2}\oint (x\,dy - y\,dx) = A,$ the area enclosed by the curve.
2	Show that any solution of the equation $\nabla \times (\nabla \times \mathbf{A}) - k^2\mathbf{A} = 0$ automatically satisfies the vector Helmholtz equation $\nabla^2\mathbf{A} + k^2\mathbf{A} = 0$ and the solenoidal condition $\nabla \cdot \mathbf{A} = 0$.
3	The vector $\mathbf{r}$, starting at the origin, terminates at and specifies the point in space (x, y, z), if $(\mathbf{r} - 2\mathbf{a}) \cdot \mathbf{r} = 0$. Find the surface swept out by the tip of $\mathbf{r}$ and describe the geometric role of $\mathbf{a}$. (The vector $\mathbf{a}$ is constant in magnitude and direction).
4	In the spherical polar coordinate system, $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$. The transformation equations corresponding to $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$. Calculate the spherical polar coordinate scale factors: h_r, h_θ, and h_ϕ.

5	Represent ε_{ij} by a 2×2 matrix, and using the 2×2 rotation matrix $\boldsymbol{R}$ show that ε_{ij} is invariant under orthogonal similarity transformations.

مع اطيب التمنيات بالنجاح

Mathematical supplement

1) Gauss' theorem states that

$$\oiint_{\partial V} \mathbf{V}\cdot d\boldsymbol{\sigma} = \iiint_V \nabla\cdot\mathbf{V}\, d\tau.$$

2) Stokes' theorem

$$\oint \mathbf{V}\cdot d\boldsymbol{\lambda} = \int_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}$$

3) The scale factors may be conveniently identified by the relation

$$\frac{\partial \mathbf{r}}{\partial q_i} = h_i \hat{\mathbf{q}}_i$$

4) The BAC–CAB rule

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

5) Useful relations

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$