Final Exam

Answer the following questions:

(8 Marks)

1

The vertices A, B, and C of a triangle are given by the points $(-1, 0, 2)$, $(0, 1, 0)$, and $(1, -1, 0)$, respectively. Find point D so that the figure $ABCD$ forms a plane parallelogram.

(8 Marks)

2

Show that

$$\frac{1}{3}\oiint_S \mathbf{r}\cdot d\sigma = V,$$

where V is the volume enclosed by the closed surface $S = \partial V$.

(8 Marks)

3

With $\mathbf{L} = -i\,\mathbf{r}\times\boldsymbol{\nabla}$, verify the operator identities

(a) $\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} - i\frac{\mathbf{r}\times\mathbf{L}}{r^2}$,

(b) $\mathbf{r}\nabla^2 - \boldsymbol{\nabla}\left(1 + r\frac{\partial}{\partial r}\right) = i\boldsymbol{\nabla}\times\mathbf{L}$.

(8 Marks)

4

Show that Laguerre's ODE, $xy''(x) + (1-x)y'(x) + ny(x) = 0$

may be put into self-adjoint form by multiplying by e^{-x}

and that $w(x) = e^{-x}$ is the weighting function.

(8 Marks)

5	In the spherical polar coordinate system, $q_1 = r$, $q_2 = \theta$, $q_3 = \varphi$. The transformation equations are $$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta.$$ (a) Calculate the spherical polar coordinate scale factors: h_r, h_θ, and h_φ. (b) Check your calculated scale factors by the relation $ds_i = h_i\, dq_i$.

مع اطيب التمنيات بالنجاح

Mathematical supplement

1) Gauss' theorem :

$$\oiint_{\partial V} \mathbf{V} \cdot d\sigma = \iiint_V \nabla \cdot \mathbf{V}\, d\tau.$$

2) The scale factors may be conveniently identified by the relation

$$\frac{\partial \mathbf{r}}{\partial q_i} = h_i \hat{\mathbf{q}}_i$$

3) The BAC–CAB rule

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

4) Useful relations

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$