9.4 SINGULAR POINTS

The interest in the concept of a singular point stems from its usefulness in (1) classifying ODEs and (2) investigating the feasibility of a series solution.

Second-order homogeneous differential equation (in *y) is*

$$y'' + P(x)y' + Q(x)y = 0$$

For finite values of x_0 *:*

1- If the functions *P(x) and Q(x) remain* finite at $x = x_0$, *point* $x = x_0$ *is an ordinary point.*

2- If either *P(x) or Q(x) diverges as* $x \to x0$ *but* $(x - x0)P(x)$ *and* $(x - x_0)^2 Q(x)$ *remain finite as* $x \to x_0$, *then* $x = x_0$ *is called a **regular*** singular point.

3- If *P(x) diverges faster than* $1/(x - x_0)$ *so that* $(x - x_0)P(x)$ *goes to infinity as* $x \to x_0$, or *Q(x) diverges faster than* $1/(x - x_0)^2$ *so that* $(x - x_0)^2 Q(x)$ *goes to infinity as* $x \to x_0$, *then point* $x = x_0$ *is labeled an **irregular singularity.***

For point $x \to \infty$:

set $x = 1/z$ substitute into the differential equation

$$z^4 \frac{d^2y}{dz^2} + [2z^3 - z^2 P(z^{-1})] \frac{dy}{dz} + Q(z^{-1}) y = 0.$$

The behavior at $x = \infty (z = 0)$ then depends on the behavior of the new coefficients,

$$\frac{2z - P(z^{-1})}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4},$$

Example 9.4.1

Bessel's equation is

$$x^2y'' + xy' + \left(x^2 - n^2\right)y = 0. \tag{9.79}$$

Comparing it with Eq. (9.75) we have

$$P(x) = \frac{1}{x}, \qquad Q(x) = 1 - \frac{n^2}{x^2},$$

which shows that point $x = 0$ is a regular singularity. By inspection we see that there are no other singular points in the finite range. As $x \to \infty (z \to 0)$, from Eq. (9.78) we have

the coefficients

$$\frac{2z - z}{z^2} \qquad \text{and} \qquad \frac{1 - n^2z^2}{z^4}.$$

Since the latter expression diverges as z^4, point $x = \infty$ is an irregular, or essential, singularity. ■

Table 9.4

	Equation	Regular singularity $x =$	Irregular singularity $x =$
1.	Hypergeometric $x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0.$	$0, 1, \infty$	–
2.	Legendre[a] $(1-x^2)y'' - 2xy' + l(l+1)y = 0.$	$-1, 1, \infty$	–
3.	Chebyshev $(1-x^2)y'' - xy' + n^2 y = 0.$	$-1, 1, \infty$	–
4.	Confluent hypergeometric $xy'' + (c-x)y' - ay = 0.$	0	∞
5.	Bessel $x^2 y'' + xy' + (x^2 - n^2)y = 0.$	0	∞
6.	Laguerre[a] $xy'' + (1-x)y' + ay = 0.$	0	∞
7.	Simple harmonic oscillator $y'' + \omega^2 y = 0.$	–	∞
8.	Hermite $y'' - 2xy' + 2\alpha y = 0.$	–	∞

[a] The associated equations have the same singular points.

The hypergeometric equation, with regular is taken as the standard form. The solutions of the other equations may then be expressed in terms of its solutions, the hypergeometric functions.

9.5 SERIES SOLUTIONS—FROBENIUS' METHOD

The method, a series expansion, will always work to obtaining one solution of the linear, second-order, homogeneous ODE, provided the point of expansion is no worse than a regular singular point.

A **linear, second-order, homogeneous ODE**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

the **most general solution may be written as**

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

The constants c_1 *and* c_2 *will eventually be fixed by boundary conditions.*

To illustrate, we apply the method the linear oscillator equation :

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \qquad \text{with known solutions } y = \sin\omega x, \cos\omega x$$

Now by supposing the solution in the form:

$$y(x) = x^k(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \qquad a_0 \neq 0,$$

The task now to find the exponent k and all the coefficients aλ.

1- By differentiating twice and substituting into the oscillator equation

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} + \omega^2 \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0. \qquad (9.86)$$

2- the coefficients of each power of *x on the left-hand side of Eq. (9.86) must vanish individually.*

Starting with the coefficient of the lowest power of *x*, for *λ = 0* , The requirement that the coefficient vanish yields **indicial**

Equation

$$a_0 k(k-1) = 0. \qquad , a_0 \neq 0$$

we must require either that *k = 0 or k = 1*

The remaining coefficients vanish yields two-term **recurrence relation**, (the coefficient of *xk+j (j ≥ 0), of Eq. (9.86). We set λ = j +2*

in the first summation and *λ = j in the second)*

$$a_{j+2}(k+j+2)(k+j+1) + \omega^2 a_j = 0$$

or

$$a_{j+2} = -a_j \frac{\omega^2}{(k+j+2)(k+j+1)}.$$

3- From the indicial equation k=0 or k=1

we first try the solution *k = 0,* The recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)},$$

which leads to

$$a_2 = -a_0 \frac{\omega^2}{1 \cdot 2} = -\frac{\omega^2}{2!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{3 \cdot 4} = +\frac{\omega^4}{4!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{5 \cdot 6} = -\frac{\omega^6}{6!} a_0,$$

and so on.

and solution is

$$y(x)_{k=0} = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots \right] = a_0 \cos \omega x.$$

If we choose the indicial equation root *k = 1,* the recurrence relation becomes

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}.$$

Substituting in $j = 0, 2, 4$, successively, we obtain

$$a_2 = -a_0 \frac{\omega^2}{2 \cdot 3} = -\frac{\omega^2}{3!} a_0,$$

$$a_4 = -a_2 \frac{\omega^2}{4 \cdot 5} = +\frac{\omega^4}{5!} a_0,$$

$$a_6 = -a_4 \frac{\omega^2}{6 \cdot 7} = -\frac{\omega^6}{7!} a_0, \qquad \text{and so on.}$$

$$y(x)_{k=1} = a_0 x \left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \cdots \right]$$

$$= \frac{a_0}{\omega} \left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots \right] = \frac{a_0}{\omega} \sin \omega x.$$

This series substitution, known as Frobenius' method.

However, there are two points about such series solutions that must be strongly emphasized:

1. The series solution should always be substituted back into the differential equation, to see if it works, as a precaution against algebraic and logical errors. If it works, it is a solution.

2. The acceptability of a series solution depends on its convergence.

Expansion About x_0

It is perfectly possible to replace Series solution with,

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda (x - x_0)^{k+\lambda}, \qquad a_0 \neq 0.$$

The point x_0 *should not be chosen at an essential singularity* (x_0 *can only be an ordinary point or regular* singular point) . The resultant series will be valid where it converges.

Symmetry of Solutions

Whenever the differential operator has a specific parity or symmetry,

$$\pm\mathcal{L}(x)y(-x) = 0,$$

$$+ \text{ if } \mathcal{L}(x) \text{ is even, } - \text{ if } \mathcal{L}(x) \text{ is odd.}$$

Then any solution may be resolved into even and odd parts,

$$y(x) = \tfrac{1}{2}\left[y(x) + y(-x)\right] + \tfrac{1}{2}\left[y(x) - y(-x)\right],$$

- Legendre, Chebyshev, Bessel, simple harmonic oscillator, and Hermite equations all exhibit this even parity. *Solutions of all of them* may be presented as series of even powers of *x and separate series of odd powers of x.*
- The Laguerre differential operator has neither even nor odd symmetry; hence its solutions cannot be expected to exhibit even or odd parity

Limitations of Series Approach—Bessel's Equation

try to solve Bessel's equation,

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

Again, assuming a solution of the form

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

the **indicial equation**

$$k^2 - n^2 = 0$$

with solutions $k = \pm n$.

the recurrence relation

$$a_{j+2} = -a_j \frac{1}{(j+2)(2n+j+2)}$$

The final solution take the form:

$$y(x) = a_0 2^n n! \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(n+j)!} \left(\frac{x}{2}\right)^{n+2j}$$

Notice that

When *k =n* , this solution identified as the Bessel function *Jn(x), which has either even or odd symmetry.*

When *k =−n,* and *n is not an integer, we may generate a second distinct series, to be* labeled *J−n(x).*

But if (n) an integer, the second solution simply reproduces the first.

$$J_{-n}(x) = (-1)^n J_n(x)$$

We have failed to construct a second independent solution for Bessel's equation by this series technique when *n is an integer*

This method of series solution will not always work.

Regular and Irregular Singularities :

The success of the series substitution method depends on the roots of the indicial equation and the degree of singularity of the coefficients in the differential equation.

So that, the method works with a regular singularity but failed when we had the irregular singularity. Also the method fail when the series solution diverges.

Fuchs' Theorem

which asserts that we can always obtain at least one power-series solution, provided we are expanding about a point that is an ordinary point or at worst a regular singular point.

9.6 A SECOND SOLUTION

Linear Independence of Solutions

Given a set of functions ϕ_λ, *the criterion for linear dependence is the existence of a relation* of the form

$$\sum_\lambda k_\lambda \varphi_\lambda = 0$$

in which not all the coefficients *kλ are zero, but,* if the only solution is $k_\lambda = 0$ *for all λ, the set of functions ϕ_λ is said to be linearly* independent.

By assuming that the functions *ϕλ are differentiable as needed, we can construct* **The Wronskian** determinant,

$$W = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

If W≠0 The set of functions ϕ_λ *is therefore linearly independent*

If W=0 over the entire range of the variable, the functions *ϕλ are* linearly dependent over this range

Example 9.6.1 LINEAR INDEPENDENCE

The solutions of the linear oscillator equation (9.84) are $\varphi_1 = \sin \omega x$, $\varphi_2 = \cos \omega x$. The Wronskian becomes

$$\begin{vmatrix} \sin \omega x & \cos \omega x \\ \omega \cos \omega x & -\omega \sin \omega x \end{vmatrix} = -\omega \neq 0.$$

These two solutions, φ_1 and φ_2, are therefore linearly independent. For just two functions this means that one is not a multiple of the other, which is obviously true in this case.

You know that

$$\sin \omega x = \pm\left(1 - \cos^2 \omega x\right)^{1/2},$$

but this is **not** a **linear** relation, of the form of Eq. (9.111). ■

Examples 9.6.2 LINEAR DEPENDENCE

For an illustration of linear dependence, consider the solutions of the one-dimensional diffusion equation. We have $\varphi_1 = e^x$ and $\varphi_2 = e^{-x}$, and we add $\varphi_3 = \cosh x$, also a solution. The Wronskian is

$$\begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = 0.$$

The determinant vanishes for all x because the first and third rows are identical. Hence e^x, e^{-x}, and $\cosh x$ are linearly dependent, and, indeed, we have a relation of the form of Eq. (9.111):

$$e^x + e^{-x} - 2\cosh x = 0 \qquad \text{with } k_\lambda \neq 0.$$

■

A Second Solution:

Theorem : a second-order homogeneous ODE has two linearly independent solutions.

There is no guarantee that, the power series solution of a second-order homogeneous ODE will yield the two independent solutions.

Ther e are two methods of obtaining such a second independent solution:

1- an integral method

2- power series containing a logarithmic term.

1- an integral method

Returning to our linear, second-order, homogeneous ODE of the general form

$$y'' + P(x)y' + Q(x)y = 0,$$

For the general case, let us now assume that we have one solution by a series substitution, a second independent Solution can be written as

$$y_2(x) = y_1(x) \int_b^x \frac{\exp[-\int_a^{x_2} P(x_1)dx_1]}{[y_1(x_2)]^2} dx_2.$$

a, b are arbitrary constants. If we have the **important special case** of *P(x) = 0,*

$$y_2(x) = y_1(x) \int^x \frac{dx_2}{[y_1(x_2)]^2}.$$

This means that, we can take one known solution and by integrating can generate a second, independent solution

Example 9.6.3 A SECOND SOLUTION FOR THE LINEAR OSCILLATOR EQUATION

From $d^2y/dx^2 + y = 0$ with $P(x) = 0$ let one solution be $y_1 = \sin x$. By applying Eq. (9.128), we obtain

$$y_2(x) = \sin x \int^x \frac{dx_2}{\sin^2 x_2} = \sin x(-\cot x) = -\cos x,$$

which is clearly independent (not a linear multiple) of $\sin x$. ■

2- power series containing a logarithmic term:

The second solution of our differential equation may be obtained by the following sequence of operations.

$$y'' + P(x)y' + Q(x)y = 0,$$

1- Express *P(x) and Q(x) as*

$$P(x) = \sum_{i=-1}^{\infty} p_i x^i, \qquad Q(x) = \sum_{j=-2}^{\infty} q_j x^j.$$

2. Develop the first few terms of a power-series solution,

3. Using this solution as y_1, *obtain a second series type solution,* y_2, *with Eq.*

$$y_2(x) = y_1(x) \int_b^x \frac{\exp[-\int_a^{x_2} P(x_1)dx_1]}{[y_1(x_2)]^2} dx_2.$$

And integrating term by term,

we obtain

$$y_2(x) = y_1(x) \int^x \left(c_0 x_2^{-n-1} + c_1 x_2^{-n} + c_2 x_2^{-n+1} + \cdots + c_n x_2^{-1} + \cdots \right) dx_2.$$

where *n is zero* or a positive integer

The integration indicated in leads to a coefficient of $y_1(x)$ *consisting of two* parts:

- A power series starting with x^{-n}.
- A logarithm term from the integration of x^{-1}

Exercises:

9.5.13

9.5.14

9.5.17