9.3 SEPARATION OF VARIABLES

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- Our first technique for solution, splits the partial differential equation of *n variables* into *n ordinary differential equations.*

- *Each separation introduces an arbitrary constant*

of separation.

- If we have *n variables, we have to introduce n−1 constants, determined* by the conditions imposed in the problem being solved.

we manipulate the equation so it was in the form

$$f(y)\,dy = g(x)\,dx.$$

In other words, we separated x and y so each variable had its own side, including the dx and the dy that formed the derivative expression $\frac{dy}{dx}$. This is why the method is called "separation of variables."

Not all differential equations are like that. For example, $\frac{dy}{dx} = x + y$ cannot be brought to the form $f(y)\,dy = g(x)\,dx$ no matter how much we try.

In fact, a major challenge with using separation of variables is to identify where this method is applicable. Differential equations that can be solved using separation of variables are called **separable equations**.

Cartesian Coordinates :

In Cartesian coordinates the Helmholtz equation becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

The equation separable and we can replace the function by product of three functions

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

Substitute

$$YZ\frac{d^2 X}{dx^2} + XZ\frac{d^2 Y}{dy^2} + XY\frac{d^2 Z}{dz^2} + k^2 XYZ = 0.$$

Dividing by ψ = *XYZ and rearranging terms*

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2 Y}{dy^2} - \frac{1}{Z}\frac{d^2 Z}{dz^2}$$

x, y, and z are all independent coordinates. Therefore, each side must be equal to a constant, a constant of separation

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2,$$

$$-k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} = -l^2$$

second separation

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 + l^2 - \frac{1}{Z}\frac{d^2Z}{dz^2}$$

by equating each side to another constant of separation, $-m^2$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2,$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 + l^2 + m^2 = -n^2$$

solution should be labeled according to the choice of our constants *l,m, and n;*

$$\psi_{lm}(x, y, z) = X_l(x)Y_m(y)Z_n(z)$$

Subject to the conditions of the problem and to the condition $k^2 = l^2 + m^2 + n^2$, *we may choose l, m, and n as we like*

the most general solution may be taken by taking a linear combination of solutions ψ_{lm} ,

$$\Psi = \sum_{l,m} a_{lm}\psi_{lm}.$$

The constant coefficients a_{lm} *are finally chosen to permit to satisfy the boundary conditions* of the problem, which, as a rule, lead to a discrete set of values *l,m.*

Circular Cylindrical Coordinates :

unknown function ψ *dependent on* ρ, ϕ, *and z, the Helmholtz equation becomes*

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0$$

assume a factored form for ψ,

$$\psi(\rho,\varphi,z)=P(\rho)\Phi(\varphi)Z(z).$$

Substituting

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{PZ}{\rho^2}\frac{d^2\Phi}{d\varphi^2}+P\Phi\frac{d^2Z}{dz^2}+k^2P\Phi Z=0$$

Moving the *z derivative to the right-hand side yields*

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}+k^2=-\frac{1}{Z}\frac{d^2Z}{dz^2}.$$

each side equal to the same constant

$$\frac{d^2 Z}{dz^2} = l^2 Z$$

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2} + k^2 = -l^2.$$

Setting $k^2 + l^2 = n^2$, *multiplying by* ρ^2, *and rearranging terms*

$$\frac{\rho}{P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + n^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}.$$

In the second separaion the right-hand side to m^2

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi.$$

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + (n^2\rho^2 - m^2)P = 0$$

the most general solution of the Helmholtz equation is a linear combination of the product solutions:

$$\psi(\rho, \varphi, z) = P(\rho)\Phi(\varphi)Z(z).$$

$$\Psi(\rho, \varphi, z) = \sum_{m,n} a_{mn} P_{mn}(\rho)\Phi_m(\varphi)Z_n(z)$$

Spherical Polar Coordinates :

Helmholtz equation, again with k^2 *constant, in spherical polar* coordinates.

$$\frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \varphi^2}\right] = -k^2 \psi$$

$$\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$$

dividing by *RΘΦ :*

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi r^2 \sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = -k^2$$

By multiplying by $r^2 \sin^2\theta$, we can isolate $(1/\Phi)(d^2\Phi/d\varphi^2)$ to obtain

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = r^2 \sin^2\theta\left[-k^2 - \frac{1}{r^2 R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{1}{r^2 \sin\theta\,\Theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)\right]$$

let us use $-m^2$ *as the separation constant*

$$\frac{1}{\Phi}\frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2$$

$$\frac{1}{r^2 R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{r^2\sin^2\theta} = -k^2$$

Multiplying by *r2 and rearranging terms*

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + r^2k^2 = -\frac{1}{\sin\theta\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin^2\theta}$$

Again, the variables are separated.We equate each side to a constant, *Q,*

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0,$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{QR}{r^2} = 0.$$

general solution may be written

$$\psi_{Qm}(r,\theta,\varphi) = \sum_{Q,m} a_{Qm} R_Q(r)\Theta_{Qm}(\theta)\Phi_m(\varphi)$$

Table 9.2 Solutions in Spherical Polar Coordinates[a]

$$\psi = \sum_{l,m} a_{lm} \psi_{lm}$$

1.	$\nabla^2 \psi = 0$	$\psi_{lm} = \begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$ [b]
2.	$\nabla^2 \psi + k^2 \psi = 0$	$\psi_{lm} = \begin{Bmatrix} j_l(kr) \\ n_l(kr) \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$ [b]
3.	$\nabla^2 \psi - k^2 \psi = 0$	$\psi_{lm} = \begin{Bmatrix} i_l(kr) \\ k_l(kr) \end{Bmatrix} \begin{Bmatrix} P_l^m(\cos\theta) \\ Q_l^m(\cos\theta) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$ [b]

Table 9.3 Solutions in Circular Cylindrical Coordinates[a]

$$\psi = \sum_{m,\alpha} a_{m\alpha} \psi_{m\alpha}$$

a.	$\nabla^2 \psi + \alpha^2 \psi = 0$	$\psi_{m\alpha} = \begin{Bmatrix} J_m(\alpha\rho) \\ N_m(\alpha\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} e^{-\alpha z} \\ e^{\alpha z} \end{Bmatrix}$
b.	$\nabla^2 \psi - \alpha^2 \psi = 0$	$\psi_{m\alpha} = \begin{Bmatrix} I_m(\alpha\rho) \\ K_m(\alpha\rho) \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix} \begin{Bmatrix} \cos \alpha z \\ \sin \alpha z \end{Bmatrix}$
c.	$\nabla^2 \psi = 0$	$\psi_m = \begin{Bmatrix} \rho^m \\ \rho^{-m} \end{Bmatrix} \begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}$

Exercises

9.2.4

9.2.12

9.2.13

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