CHAPTER 9 - **DIFFERENTIAL EQUATIONS**

9.1 PARTIAL DIFFERENTIAL EQUATIONS

- Almost all the elementary and numerous advanced parts of theoretical physics are formulated in terms of differential equations, specially second order differential equations.
- Differential equations in one variable (abbreviated **ODEs),**

differential equations **in two or more variables** (abbreviated **PDEs).**

- Thus, ODEs and PDEs appear as linear operator equations,

$$L\psi = F,$$

where *F is a known (source) function,*

L is a linear combination of derivatives,

and *ψ is the unknown function or solution.*

- Any linear combination of solutions is again a solution if $F = 0$ *(*homogeneous PDEs.*)*

Examples of PDEs

1. Laplace's equation, $\nabla^2\psi = 0$.
2. Poisson's equation, $\nabla^2\psi = -\rho/\varepsilon 0$.
3. time-independent and time-dependent diffusion equations

$$\nabla^2\psi \pm k^2\psi = 0, \qquad \partial^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

5. The time-dependent wave equation, $\partial^2\psi = 0$.
6. The Klein–Gordon equation, $\partial^2\psi = -\mu^2\psi$
8. The Schrödinger wave equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

General techniques for solving second-order PDEs

- Separation of variables, where the PDE is split into ODEs that are related by common constants .
- Conversion of a PDE into an integral equation using **Green's functions applies to inhomogeneous PDEs,**
- Other analytical methods, such as the use of integral transforms

Classes of PDEs and Characteristics

Linear **PDEs : can be represented by** a linear operator

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f,$$

which can be reduced to three classes according to whether the discriminant *D = ac − b2 > 0, = 0, or < 0.*

(i) Elliptic PDEs ∇^2 or $c^{-2}\partial^2/\partial t^2 + \nabla^2$

(ii) parabolic PDEs, $a\partial/\partial t + \nabla^2$

(iii) hyperbolic PDEs, $c^{-2}\partial^2/\partial t^2 - \nabla^2$

Nonlinear PDEs : The simplest nonlinear wave equation, results if the speed of propagation, *c, is not constant but depends on the wave ψ.*

$$\frac{\partial \psi}{\partial t} + c(\psi)\frac{\partial \psi}{\partial x} = 0.$$

Boundary Conditions

Solutions usually are required to satisfy certain conditions.

- **initial conditions**
- **Boundary conditions**

boundary conditions may take three forms:

1. **Cauchy boundary conditions. The value of a function and normal derivative specified** on the boundary
2. **Dirichlet boundary conditions. The value of a function specified on the boundary**
3. **Neumann boundary conditions. The normal derivative (normal gradient) of a function** specified on the boundary.

A summary of the relation of these three types of boundary conditions to the three types of two-dimensional partial differential equations is given in Table

Boundary conditions	Type of partial differential equation		
	Elliptic	Hyperbolic	Parabolic
	Laplace, Poisson in (x, y)	Wave equation in (x, t)	Diffusion equation in (x, t)
Cauchy			
Open surface	Unphysical results (instability)	**Unique, stable solution**	Too restrictive
Closed surface	Too restrictive	Too restrictive	Too restrictive
Dirichlet			
Open surface	Insufficient	Insufficient	**Unique, stable solution in one** direction
Closed surface	**Unique, stable solution**	Solution not unique	Too restrictive
Neumann			
Open surface	Insufficient	Insufficient	**Unique, stable solution in one** direction
Closed surface	**Unique, stable solution**	Solution not unique	Too restrictive

9.2 FIRST-ORDER DIFFERENTIAL EQUATIONS

consider here differential equations of the general form

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x, y)}{Q(x, y)}. \qquad (9.16)$$

It is **first order ordinary differential equation** *, it may or may not be* ***linear***

Separable Variables

the special form of Eq. (9.16)

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x)}{Q(y)}.$$

$$P(x)\,dx + Q(y)\,dy = 0.$$

$$\int_{x_0}^{x} P(x)\,dx + \int_{y_0}^{y} Q(y)\,dy = 0.$$

Note that this separation of variables technique does not require that the differential equation be linear.

Example 9.2.1

We want to find the velocity of the falling parachutist as a function of time

For simplicity we assume that the parachute opens immediately, that is, at time $t = 0$, where $v(t = 0) = 0$, our initial condition. Newton's law applied to the falling parachutist gives

$$m\dot{v} = mg - bv^2,$$

The terminal velocity, v_0, can be found from the equation of motion as $t \to \infty$; when there is no acceleration, $\dot{v} = 0$, so

$$bv_0^2 = mg, \qquad \text{or} \qquad v_0 = \sqrt{\frac{mg}{b}}.$$

The variables t and v separate

$$\frac{dv}{g - \frac{b}{m}v^2} = dt,$$

which we integrate by decomposing the denominator into partial fractions. The roots of the denominator are at $v = \pm v_0$. Hence

$$\left(g - \frac{b}{m}v^2\right)^{-1} = \frac{m}{2v_0 b}\left(\frac{1}{v + v_0} - \frac{1}{v - v_0}\right).$$

Integrating both terms yields

$$\int^{v} \frac{dV}{g - \frac{b}{m}V^2} = \frac{1}{2}\sqrt{\frac{m}{gb}}\ln\frac{v_0 + v}{v_0 - v} = t.$$

Solving for the velocity yields

$$v = \frac{e^{2t/T} - 1}{e^{2t/T} + 1}v_0 = v_0\frac{\sinh\frac{t}{T}}{\cosh\frac{t}{T}} = v_0 \tanh\frac{t}{T},$$

where $T = \sqrt{\frac{m}{gb}}$ is the time constant governing the asymptotic approach of the velocity to the limiting velocity, v_0.

Putting in numerical values, $g = 9.8\ \mathrm{m/s^2}$ and taking $b = 700\ \mathrm{kg/m}$, $m = 70\ \mathrm{kg}$, gives $v_0 = \sqrt{9.8/10} \sim 1\ \mathrm{m/s} \sim 3.6\ \mathrm{km/h} \sim 2.23\ \mathrm{mi/h}$, the walking speed of a pedestrian at landing, and $T = \sqrt{\frac{m}{bg}} = 1/\sqrt{10 \cdot 9.8} \sim 0.1\ \mathrm{s}$. Thus, the constant speed v_0 is reached within a second. Finally, because **it is always important to check the solution**, we verify that our solution satisfies

$$\dot{v} = \frac{\cosh t/T}{\cosh t/T}\frac{v_0}{T} - \frac{\sinh^2 t/T}{\cosh^2 t/T}\frac{v_0}{T} = \frac{v_0}{T} - \frac{v^2}{T v_0} = g - \frac{b}{m}v^2,$$

that is, Newton's equation of motion. The more realistic case, where the parachutist is in free fall with an initial speed $v_i = v(0) > 0$ before the parachute opens, is addressed in Exercise 9.2.18. ■

Exact Differential Equations

This equation is said to be **exact if can find** function $\phi(x, y)$ = constant and $d\phi = 0$:

$$P(x, y)\,dx + Q(x, y)\,dy = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

and

$$\frac{\partial \varphi}{\partial x} = P(x, y), \qquad \frac{\partial \varphi}{\partial y} = Q(x, y).$$

The necessary and sufficient condition for our equation to be exact is

$$\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x} = \frac{\partial^2 \varphi}{\partial x \partial y}.$$

If $\phi(x, y)$ *exists, then the solution is*

$$\phi(x, y) = C.$$

- It may well turn out that Eq. (9.16) is not exact and the previous condition is not satisfied.
- In this case, there always exists at least one or more of integrating factors *α(x, y) such that.*

$$\alpha(x, y)P(x, y)\,dx + \alpha(x, y)Q(x, y)\,dy = 0$$

is exact.

Unfortunately, an integrating factor is not always obvious or easy to find.

- A differential equation in which the variables have been separated is automatically exact.
- An exact differential equation is not necessarily separable.

Linear First-Order ODEs

•If *f (x, y) in Eq. (9.16) has the form −p(x)y + q(x), then Eq. (9.16) becomes;*

$$\frac{dy}{dx} + p(x)y = q(x). \qquad (9.25)$$

•Equation (9.25) is the most general **linear first-order ODE;** the linearity refers to the *y and dy/dx. (*There are no higher powers, that is, y^2*, and no products, y(dy/dx)).*

•Equation (9.25) may be solved exactly; the complete general solution of the linear, first-order differential is

$$y(x) = \exp\left[-\int^x p(t)\,dt\right]\left\{\int^x \exp\left[\int^s p(t)\,dt\right]q(s)\,ds + C\right\}.$$

With integrating factor *α(x)*

$$\alpha(x) = \exp\left[\int^x p(x)\,dx\right]$$

Example 9.2.2 RL CIRCUIT

For a resistance-inductance circuit Kirchhoff's law leads to

$$\boxed{L\frac{dI(t)}{dt} + RI(t) = V(t)}$$

for the current $I(t)$, where L is the inductance and R is the resistance, both constant. $V(t)$ is the time-dependent input voltage.

From Eq. (9.29) our integrating factor $\alpha(t)$ is

$$\alpha(t) = \exp\int^t \frac{R}{L}dt = e^{Rt/L}.$$

Then by Eq. (9.30),

$$I(t) = e^{-Rt/L}\left[\int^t e^{Rt/L}\frac{V(t)}{L}dt + C\right],$$

with the constant C to be determined by an initial condition (a boundary condition).

For the special case $V(t) = V_0$, a constant,

$$I(t) = e^{-Rt/L}\left[\frac{V_0}{L}\cdot\frac{L}{R}e^{Rt/L} + C\right] = \frac{V_0}{R} + Ce^{-Rt/L}.$$

If the initial condition is $I(0) = 0$, then $C = -V_0/R$ and

$$\boxed{I(t) = \frac{V_0}{R}[1 - e^{-Rt/L}].}$$

■